Do three questions from each section. If you work more than the required number of problems, make sure that you clearly mark which problems you want to have counted. Standard results from the courses may be used without proof provided they are clearly stated. If you have doubts about the wording of a problem or about what results may be assumed without proof, please ask for clarification. In no case should you interpret a problem in such a way that it becomes trivial.

Section A.

Question 1. Prove that if $f : X \to Y$ is continuous and surjective, X is compact, and Y is Hausdorff, then f is a quotient map.

Question 2.
a. Define what it means for a topological space to be normal.
b. Prove that a closed subspace of a normal space is normal.
c. Suppose that X is a normal topological space, that $C \subseteq X$ is closed, and that $C \subseteq O_1 \cup O_2$, where O_1, O_2 are open. Prove that there exist closed sets $C_1, C_2 \subseteq X$ with $C = C_1 \cup C_2$ and $C_i \subseteq O_i$, $i = 1, 2$.

Question 3. Given an integer n we say that 2^i *exactly divides* n if 2^i divides n and 2^{i+1} does not. In this case we write $2^i \mid\mid n$. Define a metric on $\mathbb{Z}$ by

$$d_2(n, m) = \begin{cases} 0 & \text{if } n = m \\ 2^{-i} & \text{if } 2^i \mid\mid (n - m). \end{cases}$$

a. Prove that d_2 is a metric. (Hereafter we let X be the metric space $(\mathbb{Z}, d_2)$.)
b. Prove that X is not discrete.
c. Prove that X is not connected.

Question 4. A topological space X is called *locally connected* if for every $x \in X$ and O a neighborhood of x there exists a connected open set V with $x \in V \subseteq O$. Prove that a compact, locally connected space has a finite number of components.

Question 5.
a. Recall that a topological space is called *seperable* if it contains a countable dense subset. Prove that a countable product of seperable spaces (with the product topology) is seperable.
b. Prove that the result in the first part is false if the product is given the box topology.

Section B.

Question 6. Define the topological space $X = (I \times S^1)/\sim$ where $\sim$ is the equivalence relation generated by $(1, z) \sim (0, z^5)$ for all $z \in S^1$ (where we think of S^1 as the unit circle in $\mathbb{C}$). Give a presentation for the fundamental group of X, carefully justifying your calculation.

Question 7. Let $f : S^2 \to S^2$ be such that $f(x) \neq f(-x)$ for all $x \in S^2$. Prove that f is surjective. [Hint: Degree theory will probably not be helpful.]

Question 8. Use covering space theory to prove that every map $\mathbb{R}P^2 \to S^1 \times S^1$ is homotopic to a constant map.

Question 9. Let X be a topological space and let $U_1 \subseteq U_2 \subseteq U_3 \subseteq \ldots$ be an increasing open cover of X. Prove that for $n \geq 0$ the singular chain groups of X satisfy

$$C_n(X) = \bigcup_{j=1}^{\infty} C_n(U_j).$$

[To be completely explicit, on the right we really mean $(i_j)_\sharp\big(C_n(U_j)\big)$ where $i_j : U_j \to X$ is the inclusion map.]

Question 10. The *suspension* SX of a topological space X is the space $(X \times I)/\sim$ where $\sim$ is the equivalence relation generated by $(0, x) \sim (0, x')$ and $(1, x) \sim (1, x')$ for all $x, x' \in X$. Compute the homology groups of SX in terms of those of X.