Do three of the problems from section A and three questions from section B. If you work more than the required number of problems, make sure that you clearly mark which problems you want to have counted. If you have doubts about the wording of a problem or about what results may be assumed without proof, please ask for clarification. In no case should you interpret a problem in such a way that it becomes trivial.

Section A.

Question 1. Let A be a subset of a topological space X and let B be a subset of a topological space Y. Let $X \times Y$ be the product space, and let $\mathrm{int}_Z(C)$ denote the interior of the set C in the space Z. Prove or give a counterexample to $\mathrm{int}_{X\times Y}(A \times B) = \mathrm{int}_X(A) \times \mathrm{int}_Y(B)$.

Question 2. Let $p : X \to Y$ be a quotient map. Show that if Y is connected and moreover each set $p^{-1}(\{y\})$ is connected, then X is connected.

Question 3.
a. Prove that a map $f : X \to Y$ between topological spaces is continuous if and only if $f(\overline{A}) \subseteq \overline{f(A)}$ for all $A \subseteq X$. [Here $\overline{A}$ denotes the closure of A.]
b. Prove that if f is continuous and $f(\overline{A})$ is closed for some $A \subseteq X$ then $f(\overline{A}) = \overline{f(A)}$.

Question 4. Let $Y = (0, 1) \times (0, 1)$ and let X be the set $X = Y \cup \{*\}$. Let $\mathcal{T}_1 := \{U \mid U$ is a Euclidean open subset of $Y\}$, let $\mathcal{T}_2 := \{X - C \mid C$ is a compact subset of the Euclidean space $Y\}$, and let $\mathcal{T} := \mathcal{T}_1 \cup \mathcal{T}_2$. Then $(X, \mathcal{T})$ is a topological space called the *one-point compactification* of $(0, 1) \times (0, 1)$ (you don't need to prove this). Show that the space $(X, \mathcal{T})$ is compact.

Section B.

Question 5. Let X be the triangular parachute formed from the standard 2-simplex Δ^2 by identifying the three vertices with one another.
a. Compute a presentation for $\pi_1(X)$.
b. Show that $\pi_1(X)$ is isomorphic to a free group F_n for some n; what is n?

Question 6. Let $p_i : (\widetilde{X}_i, \tilde{x}_i) \to (X_i, x_i)$ be covering spaces for $i = 1, 2$.
a. Show that the product space $\widetilde{X}_1 \times \widetilde{X}_2$ together with the map $p : \widetilde{X}_1 \times \widetilde{X}_2 \to X_1 \times X_2$ defined by $p(y, z) := (p_1(y), p_2(z))$ is also a covering space.
b. Find the universal covering of the space $S^1 \times D^2 \times S^1$.

Question 7. Let Y be a Δ-complex.
a. Prove that if $H_4(Y) \neq 0$, then Y must have a simplex of dimension 4.
b. Prove that if $H_4(Y) = \mathbb{Z}/7\mathbb{Z}$, then Y also must have a simplex of dimension 5.

Question 8. Let X be a hexagon in $\mathbb{R}^2$. Define an equivalence relation on X corresponding to labeling the 6 edges in the boundary of X in a counterclockwise fashion in order by: counterclockwise a, counterclockwise b, counterclockwise a, counterclockwise b, counterclockwise a, counterclockwise b. Let M be the corresponding quotient space. Compute $H_n(M)$ for all $n \geq 0$.