Solution Outlines for Chapter 10, Part A

 # 1: Prove that the mapping given in Example 2 is a homomorphism.

Let $\phi : GL(2,\mathbb{R}) \to \mathbb{R}^*$ be defined by $A \mapsto det(A)$. Let $A \in GL(2,\mathbb{R})$. This means that A is invertible thus the $det(A)$ is not zero, hence the $det(A)$ is in $\mathbb{R}^*$. So ϕ maps to $\mathbb{R}^*$ as claimed. Now let $A, B \in GL(2,\mathbb{R})$. Then $\phi(AB) = det(AB) = det(A)det(B) = \phi(A)\phi(B)$, so ϕ is a homomorphism.

 # 2: Prove that the mapping given in Example 3 is a homomorphism.

Let $\phi : \mathbb{R}^* \to \mathbb{R}^*$ be defined by $x \mapsto |x|$. Then $\phi(xy) = |xy| = |x||y| = \phi(x)\phi(y)$ so ϕ is a homomorphism.

 # 3: Prove that the mapping given in Example 4 is a homomorphism.

Let $\phi : \mathbb{R}[x] \to \mathbb{R}[x]$ be defined by $f \mapsto f'$. Then for $f, g \in \mathbb{R}[x]$, $\phi(f+g) = (f+g)' = f' + g' = \phi(f) + \phi(g)$ so ϕ is a homomorphism.

 # 6: Let G be the group of all polynomials with real coefficients under addition. For each f in G let $\int f$ denote the antiderivative of f that passes through the point $(0,0)$. Show that the mapping $f \mapsto \int f$ from G to G is a homomorphism. What is the kernel of this mapping? Is this mapping a homomorphism if $\int f$ denotes the antiderivative of f that passes through $(0,1)$?

Let $\phi : \mathbb{R}[x] \to \mathbb{R}[x]$ be defined by $f \mapsto \int f$. Then $\phi(f+g) = \int(f+g) + c$ where $c = -(f+g)(0)$ and $\int(f+g)$ is the polynomial that is the antiderivative without a constant term. Now, $\phi(f) + \phi(g) = \int f + \int g + c_1 + c_2$ where $c_1 = -f(0)$ and $c_2 = -g(0)$. Hence $\phi(f+g) = \phi(f) + \phi(g)$ for all $f, g \in \mathbb{R}[x]$ so ϕ is a homomorphism.

Now, the kernel of ϕ are the set of things that map to the identity, 0. So $Ker\phi = \{f | \int f = 0\} = \{a_0 + a_1x + \ldots + a_nx^n \big| 0 + a_0 * x + \frac{a_1}{2}x^2 + \ldots + \frac{a_n}{n+1}x^{n+1} = 0\} = \{a_0 = a_1 = \cdots = a_n = 0\} = \{0\}$.

If the function $\int f$ goes through (0,1) instead, it is not a homomorphism. This is because $\phi(f+g)(0) = 1$ but $(\phi(f) + \phi(g))(0) = \phi(f)(0) + \phi(g)(0) = 1 + 1 = 2$, so the functions are not the same.

 # 7: If ϕ is a homomorphism from G to H and σ is a homomorphism from H to K, show that $\sigma\phi$ is a homomorphism from G to K. How are $Ker\phi$ and $Ker\sigma\phi$ related?

Let $\phi : G \to H$ be a homomorphism and $\sigma : H \to K$ also be a group homomorphism. Then $\sigma\phi : G \to K$ and $\sigma\phi(xy) = \sigma(\phi(x)\phi(y)) = \sigma(\phi(x))\sigma(\phi(y)) = \sigma\phi(x)\sigma\phi(y)$ so the composition is a homomorphism.

Notice that σ is a homomorphism so the $\ker\phi$ maps to the identity in H. Since σ is also a homomorphism, it maps this identity to the identity in K. Thus, $Ker\phi \subseteq Ker\sigma\phi$. Note that more things from H could map to the identity in K so we do not know that the $Ker\phi = Ker\sigma\phi$.

 # 10: Let G be a subgroup of some dihedral group. For each x in G define $\phi(x)$ to be $+1$ if x is a rotation and -1 if x is a reflection. Prove that ϕ is a homomorphism from G to the multiplicative group $\{+1,\ -1\}$. What is the kernel?

Let ϕ be defined as above. Now, elements in G either look like a rotation or a flip. So all elements are either of the form r^i or $r^i f$. Thus to show that ϕ is a homomorphism, I need to consider four cases: two rotations multiplied, two flips multiplied, a rotation times a flip, and a flip times a rotation (recall: not Abelian). We consider each in turn below and conclude that ϕ is a homomorphism.

- $\phi(r^i \circ r^j) = \phi(r^{i+j}) = 1 = 1 \cdot 1 = \phi(r^i) \cdot \phi(r^j)$
- $\phi(r^i f \circ r^j f) = \phi(r^i r^{n-j} f f) = \phi(r^{i+n-j}) = 1 = -1 \cdot -1 = \phi(r^i f) \cdot \phi(r^j f)$
- $\phi(r^i \circ r^j f) = \phi(r^{i+j} f) = -1 = 1 \cdot -1 = \phi(r^i) \cdot \phi(r^j f)$
- $\phi(r^i f \circ r^j) = \phi(r^i r^{n-j} f) = \phi(r^{i+n-j} f) = -1 = -1 \cdot 1 = \phi(r^i f) \cdot \phi(r^j)$

$Ker\phi = \{g \big| \phi(g) = 1\} = \{$ rotations in $G\} =< r >$

 # 14: Explain why the correspondence $x \mapsto 3x$ from $\mathbb{Z}_{12}$ to $\mathbb{Z}_{10}$ is not a homomorphism.

If the correspondence is a homomorphism, then it should preserve the operation. Let's show this is not true bia a counter example. Take $6, 7 \in \mathbb{Z}_{12}$. Then $\phi(6 + 7) = \phi(1) = 3$. But $\phi(6) + \phi(7) = 18 + 21 = 8 + 1 = 9$. Since $3 \neq 9$ in $\mathbb{Z}_{10}$ the operation is not preserved.