Solution Outlines for Chapter 11

 # 2: What is the smallest positive integer n such that there are three non isomorphic Abelian groups of order n? Name the three groups.

$n = 8$, $\mathbb{Z}_8$, $\mathbb{Z}_2 \oplus \mathbb{Z}_4$, $\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$

 # 4: Calculate the number of elements of order 2 in each of $\mathbb{Z}_{16}$, $\mathbb{Z}_8 \oplus \mathbb{Z}_2$, $\mathbb{Z}_4 \oplus \mathbb{Z}_4$ and $\mathbb{Z}_4 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$. Do the same for the elements of order 4.

Group	Order 2	Order 4
Z_{16}	$\phi(2) = 1$	$\phi(4) = 2$
$Z_8 \oplus \mathbb{Z}_2$	3	4
$Z_4 \oplus \mathbb{Z}_4$	3	12
$Z_4 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$	7	8

 # 5: Prove that any Abelian group of order 45 has an element of order 15. Does every Abelian group of order 45 have an element of order 9?

The Abelian groups of order 45 are, up to isomorphism, $\mathbb{Z}_{45}$ and $\mathbb{Z}_3 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_5$. There are elements of order 15 in each. For instance, take 3 and $(1, 1, 1)$ respectively. Now, $\mathbb{Z}_{45}$ has an element of order 9, namely 5. But $\mathbb{Z}_3 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_5$ does not have an element of order 9.

 # 10: Find all the Abelian groups (up to isomorphism) or order 360.

Notice: $360 = 36 * 10 = 2 * 5 * 2 * 3 * 2 * 3 = 2^3 3^2 5$. Hence the groups, up to isomorphism, are: $\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_5$, $\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_9 \oplus \mathbb{Z}_5$, $\mathbb{Z}_2 \oplus \mathbb{Z}_4 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_5$, $\mathbb{Z}_2 \oplus \mathbb{Z}_4 \oplus \mathbb{Z}_9 \oplus \mathbb{Z}_5$, $\mathbb{Z}_8 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_5$, $\mathbb{Z}_8 \oplus \mathbb{Z}_9 \oplus \mathbb{Z}_5$

 # 12: Suppose that the order of some finite Abelian group is divisible by 10. Prove that the group has a cyclic subgroup of order 10.

Let G be an Abelian group whose order is finite and divisible by 10. Then, by the corollary on P 230, G has a subgroup of order 10. But a subgroup of order 10, has an element of order 10 - namely the element isomorphic to 1 in $\mathbb{Z}_{10}$. (Note: This is clear because all Abelian groups of order 10 are isomorphic to $\mathbb{Z}_{10}$)

 # 15: How many Abelian groups (up to isomorphism) are there:

1. **of order 6?** 1

2. **of order 15?** 1

3. **of order 42?** 1

4. **of order pq where p and q are distinct primes?** 1

5. **of order pqr where p, q and r are distinct primes?** 1

6. **Generalize parts d and e.** There is a unique Abelian group of order n iff n is not divisible by the square of a prime.

 # 16: How does the number (up to isomorphism) of Abelian groups of order n compare with the number (up to isomorphism) of Abelian groups or order m where:

1. $n = 3^2$ **and** $m = 5^2$**?** Same

2. $n = 2^4$ **and** $m = 5^4$**?** Same

3. $n = p^r$ **and** $m = q^r$**, where p and$q$ are primes?** Same

4. $n = p^r$ **and** $m = p^r q$ **where p and q are distinct primes?** Same

5. $n = p^r$ **and** $m = p^r q^2$ **where p and q are distinct primes?** There are twice as many of order m as of order n

 # 21: The set $\{1, 9, 16, 22, 29, 53, 74, 79, 81\}$ is a group under multiplication modulo 91. Determine the isomorphism class of this group.

The order of the set is 9 so the group is either $\mathbb{Z}_9$ or $\mathbb{Z}_3 \oplus \mathbb{Z}_3$. The order of 9, 16, and 22 is 3 so there are more than 2 elements of order 3 in the group. Since $\mathbb{Z}_9$ only has 2 elements of order 3, it must be that the group is $\mathbb{Z}_3 \oplus \mathbb{Z}_3$.

 # 26: Let $G = \{1, 7, 17, 23, 49, 55, 65, 71\}$ under multiplication modulo 96. Express G as an external and an internal direct product of cyclic groups.

First notice that G has 8 elements so it is either isomorphic to $\mathbb{Z}_8$, $\mathbb{Z}_4 \oplus \mathbb{Z}_2$ or $\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$. The order of 1 is 1, and there are four elements of order 4 (namely: 7, 23, 55, 71). The remaining three elements (17, 49, 65) have order 2. Since no elements have order 8, and there are elements of order 4, it is clear that $G \approx \mathbb{Z}_4 \oplus \mathbb{Z}_2$ as an external direct product. As an internal direct product, G can be expressed as $< 7 > \times < 17 >$ (or you could replace 7 and 17 with any element of order 4 and 2 respectively).

 # 34: Let G be the group of all $n \times n$ diagonal matrices with ± 1 diagonal entries. What is the isomorphism class of G?

$\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \ldots \mathbb{Z}_2$, n terms