Solution Outlines for Chapter 9

 # 6: Let $H = \left\{ \begin{bmatrix} a & b \\ 0 & d \end{bmatrix} \middle| a, b, d \in \mathbb{R}, ad \neq 0 \right\}$. Is H a normal subgroup of $GL(2, \mathbb{R})$?

No; Show directly by counter example or by multiplying the general case, $\begin{bmatrix} f & g \\ h & j \end{bmatrix} \begin{bmatrix} a & b \\ 0 & d \end{bmatrix} \left(\begin{bmatrix} f & g \\ h & j \end{bmatrix} \right)^{-1}$, to see it is not contained in H.

 # 8: Viewing $< 3 >$ and $< 12 >$ as subgroups of $\mathbb{Z}$, prove that $< 3 > / < 12 >$ is isomorphic to $\mathbb{Z}_4$. Similarly, prove that $< 8 > / < 48 >$ is isomorphic to Z_6. Generalize to arbitrary integers k and n.

First, notice $< 3 >= \{\ldots - 12, -9, -6, -3, 0, 3, 6, 9, 12, \ldots\}$ and $< 12 >= \{\ldots - 24, -12, 0, 12, 24, \ldots\}$. Now $< 3 > / < 12 >$ looks like $\{-9+ < 12 >, -6+ < 12 >, -3+ < 12 >, < 12 >, 3+ < 12 >, 6+ < 12 >, 9+ < 12 >\}$ since multiples of 12 will be absorbed by $< 12 >$. Recall $aH = bH$ if and only if $b^{-1}a \in H$. Here this tells me that because $-(3) + -9 = -12$, $3+ < 12 >= -9+ < 12 >$. Similarly, $-3+ < 12 >= 9+ < 12 >$, and $-6+ < 12 >= 6+ < 12 >$. So, $< 3 > / < 12 >= \{< 12 >, 3+ < 12 >, 6+ < 12 >, 9+ < 12 >\}$. Notice that $3+ < 12 >$ has order 4 and hence generates all of $< 3 > / < 12 >$. Thus, $< 3 > / < 12 >$ is cyclic of order 4, and hence isomorphic to $\mathbb{Z}_4$.

Now, consider $< 8 > / < 48 >$. Similar to before, it is clear that this group consists of $\{< 48 >, 8+ < 48 >, 16+ < 48 >, 24+ < 48 >, 32+ < 48 >, 40+ < 48 >\}$. Notice that still similar to before $8+ < 48 >$ is a generator of the quotient group and that the group has order 48 divided by 8, or 6. Hence, it is isomorphic to $\mathbb{Z}_6$.

In general, suppose k divides n. Then $< k > / < n >$ is of the form $\{< n >, k+ < n >, 2k+ < n >, \ldots, (n-k)+ < n >\}$. This is clearly cyclic with generator $k+ < n >$ and has order $\frac{n}{k}$. Hence $< k > / < n >$ is isomorphic to $\mathbb{Z}_{\frac{n}{k}}$.

 # 11: Let $G = \mathbb{Z}_4 \oplus U(4)$, $H =< (2, 3) >$, and $K =< (2, 1) >$. Show that G/H is not isomorphic to G/K. (This shows that $H \approx K$ does not imply that $G/H \approx G/K$.)

For clarity, we write out each of the groups: $G = \{(0, 1), (1, 1), (2, 1), (3, 1), (0, 3), (1, 3), (2, 3), (3, 3)\}$, $H = \{(2, 3), (0, 1)\}$, and $K = \{(2, 1), (0, 1)\}$. Since H and K both have order 2, they are both isomorphic to $\mathbb{Z}_2$. Straight forward calculation shows,

$$G/H = \{H = (0, 1)H = (2, 3)H, (1, 1)H = (3, 3)H, (2, 1)H = (0, 3)H, (3, 1)H = (1, 3)H\}$$

and

$$G/K = \{K = (0, 1)K = (2, 1)K, (1, 1)K = (3, 1)K, (0, 3)K = (2, 3)K, (3, 3)K = (1, 3)K\}$$

. Notice that each has 4 elements as expected since 4*2=8.

Consider $(1,3)H$: $<(1,3)H> = \{(1,3)H, (2,1)H, (3,3)H, (0,1)H\} = G/H$. So, G/H is cyclic of order 4, and hence is isomorphic to $\mathbb{Z}_4$.

However, observe that G/K is not cyclic since $<(0,1)K> = \{K\}$, $<(1,1)K> = \{(1,1)K, (2,1)K\}$, $<(0,3)K> = \{(0,3)K, (0,1)K\}$ and $<(3,3)K> = \{(3,3)K, (2,1)K\}$. In fact, we recognize that this structure is the Klein-4 group, $\mathbb{Z}_2 \oplus \mathbb{Z}_2$. Hence $G/H \not\approx G/K$.

 # 13: Prove that a factor group of an Abelian group is Abelian.

Let G be an Abelian group and consider its factor group G/H, where H is normal in G. Let aH and bH be arbitrary elements of the quotient group. Then $aHbH = (ab)H = (ba)H = bHaH$ because G is Abelian. Hence the factor group is also Abelian.

 # 14: What is the order of the element $14+<8>$ in the factor group $\mathbb{Z}_{24}/<8>$?

For completeness, observe $<8> = \{8, 16, 0\}$ and $\mathbb{Z}_{24}/<8> = \{<8>, 1+<8>, 2+<8>, 3+<8>, 4+<8>, 5+<8>, 6+<8>, 7+<8>\}$. Now let's observe $14+<8>$:

$(14+<8>) + (14+<8>) = 28+<8> = 4+8$, $(14+<8>) + (4+<8>) = 18+<8> = 2+<8>$, $(14+<8>) + (2+<8> = 16+<8> = <8>$

Hence the order of $14+8$ is 4.

 # 16: Recall that $Z(D_6) = \{e, r^3\}$. What is the order of the element $rZ(D_6)$ in the factor group $D_6/Z(D_6)$?

Notice that problem 16 here is rewritten in terms of generators and relations. Now it is clear that the order of $rZ(D_6)$ is 3 since $r^3 \in Z(D_6)$.

 # 17: Let $G = \mathbb{Z}/<20>$ and $H = <4>/<20>$. List the elements of H and G/H.

Observe: $<4> = \{\ldots, -8, -4, 0, 4, 8, 12, \ldots\}$ and $<20> = \{\ldots -40, -20, 0, 20, 40, 60, \ldots\}$. Hence $H = \{<20>, 4+<20>, 8+<20>, 12+<20>, 16+<20>\} \approx \mathbb{Z}_5$.

Now notice that $G = \{<20>, 1+<20>, 2+<20>, \ldots, 19+<20>\} \approx \mathbb{Z}_{20}$. So $G/H = \{0+<20>+H, 1+<20>+H, 2+<20>+H, 3+<20>+H\} \approx \mathbb{Z}_4$.

 # 19: What is the order of the factor group $(\mathbb{Z}_{10} \oplus U(10))/<(2,9)>$?

The order of the factor group is $\frac{|\mathbb{Z}_{10} \oplus U(10)|}{|<(2,9)>} = \frac{10 \times 4}{lcm(|2|,|9|)} = \frac{40}{lcm(5,2)} = \frac{40}{10} = 4$.

 # 21: Prove that an Abelian group of order 33 is cyclic.

Let G be an Abelian group of order 33. By Theorem 9.5, there exists an element of G, say a, such that $|a| = 3$ and an element of G, say b, such that $|b| = 11$. Since G is Abelian, $(ab)^{33} = a^{33}b^{33} = e$ so the order of ab divides 33. However, it is clear $|ab|$ is not 1, 3, or 11. Hence $|ab| = 33$ so $ab \in G$ generates G, and G is cyclic.

 # 23: Determine the order of $(\mathbb{Z} \oplus \mathbb{Z})/ < (4,2) >$. Is the group cyclic?

Notice that $(1,1)+ < (4,2) >$ has infinite order [Why? Suppose it is of finite order, say n. Then $(n,n) \in < (4,2) >$ which means $(n,n) = k(4,2)$ for some k. So $k = n/4 = n/2$ or $4n = 2n$ which means $n = 2n$ so $n = 0$ since n is an integer.]. Hence the group $(\mathbb{Z} \oplus \mathbb{Z})/ < (4,2) >$ also has infinite order.

If the quotient group is cyclic, it must be isomorphic to $\mathbb{Z}$ (from previous work) so every non-identity element should have infinite order. However, $(6,3)+ < (4,2) >$ has order 2. Hence, it is not cyclic.

 # 24: The group $(\mathbb{Z}_4 \oplus \mathbb{Z}_{12})/ < (2,2) >$ is isomorphic to one of $\mathbb{Z}_8$, $\mathbb{Z}_4 \oplus \mathbb{Z}_2$, or $\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$. Determine which one by elimination.

Observe that $H =< (2,2) >= \{(2,2),(0,4),(2,6),(0,8),(2,10),(0,0)\}$ (which has order 6 as expected). Let $G = (\mathbb{Z}_4 \oplus \mathbb{Z}_{12})/ < (2,2) >$. Then
$G = \{H,(1,0)H,(0,1)H,(1,1)H,(0,2)H,(0,3)H,(3,0)H,(1,3)H\}$ and these cosets have orders 1, 2, 4, 4, 2, 4, 4, and 2 respectively. Hence, G is not cyclic and not isomorphic to $\mathbb{Z}_8$. Further, since there is an element of order 4, G is not isomorphic to $\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$. Hnece, $G \approx \mathbb{Z}_4 \oplus \mathbb{Z}_2$.

 # 25: Let $G = U(32)$ and $H = \{1,31\}$. The group G/H is isomorphic to one of $\mathbb{Z}_8$, $\mathbb{Z}_4 \oplus \mathbb{Z}_2$, or $\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$. Determine which one by elimination.

First, we know that the order of $U(32) = 2^5 - 2^4 = 16$ so G/H has order $\frac{16}{2} = 8$ as anticipated.

Consider $3H = \{3,29\} \in G/H$: $< 3H >= \{3H,9H,27H,17H,19H,25H,11H,H\}$ so the order of $3H$ is 8. Hence $G/H =< 3H >\approx \mathbb{Z}_8$.

 # 27: Let $G = U(16)$, $H = \{1,15\}$ and $K = \{1,9\}$. Are H and K isomorphic? Are G/H and G/K isomorphic?

It is obvious that $H \approx K \approx \mathbb{Z}_2$. Now, we need to check if G/H and G/K are isomorphic. We know that each has order 4 and that there are only two such groups. Consider $3H$: $< 3H >= \{3H,9H,11H,H\}$ so $3H$ generates G/H and $G/H \approx \mathbb{Z}_4$. Now observe G/K: $< K >= \{K\}, < 3K >= \{3K,K\}, < 5K >= \{5K,K\}$ and $< 7K >= \{7K,K\}$. Thus $G/K \approx \mathbb{Z}_2 \oplus \mathbb{Z}_2$. Hence $G/K \not\approx G/H$.

 # 37: Let G be a finite group and let H be a normal subgroup of G. Prove that the order of the element gH in G/H must divide the order of g in G.

Let $|g| = n$. Then $(gH)^n = g^n H = eH = H$ so $|gH|$ must divide n.

 # 38: Let H be a normal subgroup of G and let a belong to G. If the element aH has order 3 in the group G/H and $|H| = 10$, what are the possibilities for the order of a?

First, $|G| = |aH| \times |H| = 3 \times 10 = 30$. So $|a|$ divides 30. But we also know, by the previous problem, that 3 also has to divide $|a$. Hence the possible orders for a are 3, 6, 15, and 30.

 # 40: Let ϕ be an isomorphism from a group G onto a group $\bar{G}$. Prove that if H is a normal subgroup of G, then $\phi(H)$ is a normal subgroup of $\bar{G}$.

Let H be normal in G. We want to show $y\phi(H)y^{-1} \subseteq \phi(H)$ for all $y \in \bar{G} = \phi(G)$. Since $y \in \phi(G)$, there exists an $x \in G$ such that $y = \phi(x)$. Thus $y\phi(H)y^{-1} = \phi(x)\phi(H)(\phi(x))^{-1} = \phi(xHx^{-1}) = \phi(H)$ since H is normal in G, and we are done.

 # 42: An element is called a *square* if it can be expressed in the form b^2 for some b. Suppose that G is an Abelian group and H is a subgroup of G. If every element of H is a square and every element of G/H is a square, prove that every element of G is a square. Does your proof remain valid when "square" is replaced by "nth power" where n is any integer?

Let G be an Abelian group, H be a subgroup of G and every element of both H and G/H be a square. Suppose $g \in G$. Since $g \in G$, $gH \in G/H$. But all elements of G/H are squares so there exists an $aH \in G/H$ such that $gH = (aH)^2 = a^2H$. By properties of cosets, we now have that $(a^2)^{-1}g \in H$. But every element in H is a square so there exists a $b \in H$ such that $(a^2)^{-1}g = b^2$. Solving for g we see $g = a^2b^2 = (ab)^2$ since G is Abelian. But this means that g is a square. Hence every element of G is a square.

Notice that this did not depend on a property of 2 so the proof remains valid when 2 is replaced by $n \in \mathbb{Z}$.

 # 46: Show that D_{13} is isomorphic to $Inn(D_{13})$.

First, recall that $Z(D_{13}) = \{e\}$. Now, we know that $Inn(D_{13}) \approx D_{13}/Z(D_{13}) = D_{13}$.

 # 49: Suppose that G is a non-Abelian group of order p^3 where p is prime and $Z(G) \neq \{e\}$. Prove that $|Z(G)| = p$.

First recall that $Z(G)$ is normal in G. Since G is non-Abelian, $Z(G)$ does not have order p^3. Farther, since $Z(G)$ is a non-trivial subgroup, it's order is not 1 and divides p^3 so it has order p, or p^2.

Suppose that the order of $Z(G)$ is p^2. Then $|G/Z(G)| = p$ and hence the quotient group $G/Z(G)$ is cyclic. But this implies, by Theorem 9.3, that G is Abelian, which is a contradiction. Hence $|Z(G)| = p^2$.

 # 50: If $|G| = pq$ where p and q are primes that are not necessarily distinct, prove that $|Z(G)| = 1$ or pq.

Let $|G| = pq$, as above. Since $Z(G)$ is a normal subgroup of G, $|Z(G)| = 1, p, q$, or pq. If G is Abelian, $|Z(G)| = pq$.

Assume G is not Abelian. Without loss of generality, let $|Z(G)| = p$. Then $|G/Z(G)| = q$, which is prime. Hence $|G/Z(G)|$ is cyclic and G is Abelian. But this is a contradiction. Hence $|Z(G)| = 1$.

 # 51: Let N be a normal subgroup of G and let H be a subgroup of G. If N is a subgroup of H, prove that H/N is a normal subgroup of G/N if and only if H is a normal subgroup of G.

Let N be a normal subgroup of G and let H be any subgroup of G. Assume $N \subseteq H$.

"$\Rightarrow$" Let H/N be normal in G/N. Then for all $gN \in G/N$ and $hN \in H/N$, $(gN)(hN)(gN)^{-1} = (ghg^{-1})N \in H/N$. Thus $ghg^{-1}N = h'N$ for some $h'inH$. Hence $ghg^{-1} = h'n$ for some $n \in N$. But $h' \in H$ and $n \in H$ so $h'n \in H$. Hence $gHg^{-1} \subset N$. Thus H is normal in G.

"$\Leftarrow$" The argument above reverses.

 # 56: Show that the intersection of two normal subgroups of G is a normal subgroup of G. Generalize.

Let H and K be normal subgroups of G. Let $x \in H \cap K$ and $g \in G$. Since $x \in H$, gxg^{-1} is in H. Similarly, gxg^{-1} is in K. Thus gxg^{-1} is in $H \cap K$ for all $g \in G$ and $x \in H \cap K$. Thus, $H \cap K$ is normal in G. Note that in a previous chapter we showed that $H \cap K$ is a subgroup of G, which completes the proof.

 # 61: Let H be a normal subgroup of a finite group G and let $x \in G$. If $gcd(|x|, |G/H|) = 1$, show that $x \in H$.

Let $gcd(|x|, |G/H|) = 1$ as above. From an earlier problem we know that $|xH|$ must divide $|x|$, so $gcd(|xH|, |G/H|)$ must also be 1. But we also know that $|xH|$ must divide $|G/H|$ because xH is an element of this group. Hence $|xH| = 1$ so $xH = H$, which implies $x \in H$.

 # 63: If N is a normal subgroup of G and $|G/N| = m$, show that $x^m \in N$ for all x in G.

Let $x \in G$ and $|G/N| = m$. Then $x^m N = (xN)^m = (xN)^{|G/N|} = N$ so $x^m \in N$.

 # 68: Recall that a subgroup N of a group G is called characteristic if $\phi(N) = N$ for all automorphisms ϕ of G. If N is a characteristic subgroup of G, show that N is a normal subgroup of G.

Let N be a characteristic subgroup of G. Then $\phi(N) = N$ for all automorphisms of G. In particular, $\phi_g(N) = N$ when ϕ_g is the conjugation map by g. Thus $gNg^{-1} = N$ for all $g \in G$. So N is normal in G.

Team Problem Solutions for Ch 9

 # 10: Let $H = \{(1), (12)(34)\}$ in A_4.

a. **Show that H is not normal in A_4.**

We know that $(123)H = \{(123), (134)\}$ and $H(123) = \{(123), (324)\}$. These are not equal so H is not normal in A_4.

b. **Referring to the multiplication table for A_4 in Table 5.1 on page 111, show that, although $\alpha_6 H = \alpha_7 H$ and $\alpha_9 H = \alpha_{11} H$, it is not true that $\alpha_6\alpha_9 H = \alpha_7\alpha_{11} H$. Explain why this proves that the left cosets of H do not form a group under coset multiplication.**

$\alpha_6\alpha_9 H = (243)(132)H = (12)(34)H = H$ and $\alpha_7\alpha_{11}H = (142)(234)H = (14)(23)H \neq H$. This shows that multiplication is not well defined for these cosets and hence the left cosets of H do not form a group under coset multiplication. This does not surprise us since we know that normality was required for well-defined.

 # 47: Suppose that N is a normal subgroup of a finite group G and H is a subgroup of G. If $|G/N|$ is prime, prove that H is contained in N or that $NH = G$.

Let N be a normal subgroup of a finite group G, and H be any subgroup of G. Let $|G/N| = p$, a prime. Now we know that $N \subseteq NH \subseteq G$. Therefore, $p = |G : N| = |G : NH| \times |NH : N|$. Thus $|G : NH|$ is p or 1. If $|G : NH| = 1$, then $G = NH$. If $|G : NH| = p$, then $|NH : N| = 1$ so $NH = N$, which means that $H \subseteq N$.

 # 65: If G is non-Abelian, show that $Aut(G)$ is not cyclic.

Proof. Suppose not. Let $Aut(G)$ be cyclic. Then $Inn(G)$ is cyclic since $Inn(G)$ is a subgroup of $Aut(G)$ and subgroups of cyclic groups are cyclic. We know that $Inn(G) \approx G/Z(G)$ so $G/Z(G)$ must be cyclic. But this implies that G is Abelian, which is a contradiction. Thus $Aut(G)$ is not cyclic. □