## Practice Problems

1. If $p(z)$ is a polynomial of degree greater than or equal to 2, show the sum of the residues of $\frac{1}{p(z)}$ at all the zeros of $p$ must be equal to 0.

*Proof.* $p(z)$ is a polynomial, so it has finitely many zeros $b_1, b_2, ..., b_n$. Take $R > 0$ such that $R > \max_{k=1}^n |b_k|$. Then all the zeros of $p(z)$ are contained interior the contour $C_R$, the circle of radius $R$ centered at 0, oriented counter-clockwise. So then

$$\int_\gamma \frac{1}{p(z)} dz = 2\pi i \sum_{k=1}^{n} \operatorname*{Res}_{z=b_k} p(z) = 2\pi i \operatorname*{Res}_{z=0} \frac{1}{z^2} \frac{1}{p\left(\frac{1}{z}\right)}$$

Now if $p(z) = a_m z^m + a_{m-1} z^{m-1} + ... + a_0,$ with $a_m \neq 0$, then

$$\frac{1}{z^2 p\left(\frac{1}{z}\right)} = \frac{1}{z^2(a_m z^{-m} + a_{m-1} z^{-m+1} + ... + a_0)} = \frac{z^{m-2}}{a_m + a_{m-1} z + ... + a_0 z^m},$$

where $m - 2 \geq 0$ since the degree of $p$ was assumed to be at least 2.

So finally $\frac{z^{m-2}}{a_m + a_{m-1} z + ... + a_0 z^m}$ is analytic at 0, since the denominator doesn't vanish at 0 (as $a_m \neq 0$), therefore

$$\operatorname*{Res}_{z=0} \frac{1}{z^2} \frac{1}{p\left(\frac{1}{z}\right)} = 0 \Rightarrow 2\pi i \sum_{k=1}^{n} \operatorname*{Res}_{z=b_k} p(z) = 0 \Rightarrow \sum_{k=1}^{n} \operatorname*{Res}_{z=b_k} p(z) = 0$$

□

2. Show informally that if $\gamma$ is a simple closed curve traveled counterclockwise, then

$$\int_\gamma f(z) dz = -2\pi i \sum (\text{ Residues of f outside } \gamma, \text{ including } \infty)$$

*Proof.* Intuitively, $\gamma$ is a curve on the Riemann Sphere that on the one hand encloses all the residues of $f$ inside $\gamma$, but on the other hand encloses all the poles of $f$ outside $\gamma$ including $\infty$. However if $\gamma$ is positively oriented around the poles inside $\gamma$, it is negatively oriented around the poles outside $\gamma$. In symbols:

$$\int_\gamma f(z) dz - 2\pi i \sum (\text{ Residues of f inside } \gamma) = 0$$

$$\Leftrightarrow \int_\gamma f(z) dz + 2\pi i \sum (\text{ Residues of f outside } \gamma, \text{ including } \infty) = 0$$

$$\int_\gamma f(z) dz = -2\pi i \sum (\text{ Residues of f outside } \gamma, \text{ including } \infty)$$

.

Alternatively, you could apply the Theorem in section 77 (the one used in problem 1 above), to show $\operatorname{Res}_{z=\infty} f(z) = -\sum$ Residues of f in $\mathbb{C}$, from which the result follows.

□

3. Evaluate $\int_{-\infty}^{\infty} \frac{x^2}{1+x^4} dx$

*Proof.* Let $f(z) = \frac{z^2}{1+z^4}$ and $\gamma = [-R, R] \cup C_R$, the boundary of the upper semi-circle of radius $R$. Then $f$ is analytic away from the zeros of $z^4+1$, which are $z_1 = e^{\frac{i\pi}{4}}, z_2 = e^{\frac{3i\pi}{4}}, z_1 = e^{\frac{5i\pi}{4}}$, and $z_4 = e^{\frac{7i\pi}{4}}$. Only $z_1, z_2 \in \mathbb{H}$, so for $R > 1$, $z_1, z_2$ lie inside $\gamma$. By the Residue Theorem,

$$2\pi i (\operatorname*{Res}_{z=z_1} f(z) + \operatorname*{Res}_{z=z_2} f(z)) = \int_\gamma f(z)dz = \int_{C_R} f(z)dz + \int_{-R}^{R} f(z)dz.$$

And now on $C_R$, $|f(z)| \leq \frac{R^2}{R^4-1}$, for $R > 1$, by the triangle inequality. So then

$$\int_{C_R} f(z)dz \leq \pi R \frac{R^2}{R^4-1} \to 0 \text{ as } R \to \infty.$$

Therefore $2\pi i(\operatorname{Res}_{z=z_1} f(z) + \operatorname{Res}_{z=z_2} f(z)) = \lim_{R\to\infty} \int_{-R}^{R} f(z)dz = \int_{-\infty}^{\infty} \frac{x^2}{1+x^4} dx$

Finally $\operatorname{Res}_{z=z_k} f(z) = \operatorname{Res}_{z=z_k} \frac{z^2}{1+z^4} = \left.\frac{z^2}{4z^3}\right|_{z_k} = \frac{1}{4z_k} = \frac{-z_k^3}{4}$, since $z_k^4 = -1$. So now $\frac{-z_1^3}{4} + \frac{-z_2^3}{4} = \frac{-1}{4}(e^{\frac{3\pi i}{4}} + e^{\frac{9\pi i}{4}}) = \frac{-1}{4}(e^{\frac{3\pi i}{4}} + e^{\frac{9\pi i}{4}}) = \frac{-1}{4} i\sqrt{2}$.

So $\int_{-\infty}^{\infty} \frac{x^2}{1+x^4} dx = 2\pi i(\operatorname{Res}_{z=z_1} f(z) + \operatorname{Res}_{z=z_2} f(z)) = 2\pi i \frac{-1}{4} i\sqrt{2} = \pi\frac{\sqrt{2}}{2} = \frac{\pi}{\sqrt{2}}$.

□

4. Let $a \in \mathbb{R} \setminus \{0\}$. Evaluate $\int_{-\infty}^{\infty} \frac{cos(ax)}{1+x^2} dx$

*Proof.* First we address the case $a > 0$. We apply Jordan's Lemma to the function $g(z) = e^{iaz} f(z)$, where $f(z) = \frac{1}{1+z^2}$, and then we take real parts. Let $\gamma = [-R, R] \cup C_R$, as in problem 3. $g$ is holomorphic away from the poles of $f$, which are simple poles at $i$ and $-i$., where only $i$ lies in the upper half plane. So for $R > 1$ the Residue Theorem gives:

$$2\pi i \operatorname*{Res}_{z=i} g(z) = \int_\gamma g(z)dz = \int_{C_R} g(z)dz + \int_{-R}^{R} g(z)dz.$$

Since $|f(z)| \leq \frac{1}{R^2-1} \to 0$ as $R \to \infty$, Jordan's Lemma implies

$$\int_{C_R} g(z)dz \to 0 \text{ as } R \to \infty.$$

Thus, letting $R \to \infty$ above, we have

$$2\pi i \operatorname*{Res}_{z=i} g(z) = \int_{-\infty}^{\infty} \frac{e^{iax}}{1+x^2} dx.$$

And finally, $\operatorname{Res}_{z=i} g(z) = \operatorname{Res}_{z=i} e^{iaz} \frac{1}{1+z^2} = \left. e^{iaz} \frac{1}{2z} \right|_{z=i} = \frac{e^{-a}}{2i}$. So then

$$\int_{-\infty}^{\infty} \frac{e^{iax}}{1+x^2} dx = 2\pi i \operatorname*{Res}_{z=i} g(z) = 2\pi i \frac{e^{-a}}{2i} = \pi e^{-a}.$$

Taking real parts of both sides, gives us

$$\int_{-\infty}^{\infty} \frac{cos(ax)}{1+x^2} dx = \pi e^{-a}$$

When $a < 0$, a similar argument shows

$$\int_{-\infty}^{\infty} \frac{cos(ax)}{1+x^2} dx = \pi e^{a}.$$

So finally we have $\forall a \neq 0$

$$\int_{-\infty}^{\infty} \frac{cos(ax)}{1+x^2} dx = \pi e^{-|a|}$$

$\square$

5. Show that for $p > 0, q > 0$ we have

$$\int_{0}^{\infty} \frac{log(px)}{q^2+x^2} dx = \frac{\pi}{2q} log(pq)$$

*Proof.* Let $g(z) = \frac{\log(pz)}{q^2+z^2}$. Note to define $\log(pz)$ we have to define a branch cut, so let $\log(pz) = \ln|z| + iarg(z)$, where $-\frac{\pi}{2} < arg(z) < \frac{3\pi}{2}$ (i.e. with a branch along the non-positive imaginary axis). Note $\log(pz)$ is not defined at the origin, so we'll have to take our contour to be the boundary of an upper half-disk, but with a small semicircle around the origin. I.e. let $\gamma = [r, R] \cup C_R \cup [-R, -r] \cup C_r^*$, where * denotes clockwise.

Note $g$ is analytic in and on $\gamma$, (for $R$ sufficiently large, $r$ sufficiently small), except at simple poles $z = iq, -iq$, where only $iq$ lies inside $\gamma$. Furthermore, $\text{Res}_{z=qi}\, g(z) = \frac{\log(piq)}{2iq} = \frac{\ln(pq)+\frac{i\pi}{2}}{2iq}$ (using $p, q > 0$) $= \frac{\ln(pq)}{2qi} + \frac{\pi}{4q}$.

So by the Residue Theorem,

$$2\pi i\Big(\frac{\ln(pq)}{2qi} + \frac{\pi}{4q}\Big) = \int_\gamma g(z)dz = \int_r^R g(z)dz + \int_{C_R} g(z)dz + \int_{-R}^{-r} g(z)dz - \int_{C_r} g(z)dz$$

Now, $|\int_{C_R} g(z)dz| \leq \frac{\ln(R)+\pi}{R^2-q^2} \to 0$ as $R \to \infty$.
And $|\int_{C_r} g(z)dz| \leq \frac{|\ln(r)|+\pi}{q^2-r^2} \to 0$ as $r \to 0$.
Further, $\int_r^R g(z)dz \to \int_0^\infty \frac{log(px)}{q^2+x^2}$
And $\int_{-R}^{-r} g(z)dz \to \int_0^\infty \frac{log(px)}{q^2+x^2} + \pi i \int_0^\infty \frac{1}{q^2+x^2}$ (using the subsitution $\tilde{x} = -x$)

Take real and imaginary parts above to conclude

$$\frac{\pi \ln(pq)}{q} = 2\int_0^\infty \frac{log(px)}{q^2+x^2} \Rightarrow \int_0^\infty \frac{log(px)}{q^2+x^2} = \frac{\pi \ln(pq)}{2q}$$

$\square$

6. How many zeroes does $z^4 - 5z + 1$ have in $\{1 < |z| < 2\}$? (Note: this is not the set $\{1 \le |z| < 2\}$)

*Proof.* We apply Rouche's Theorem to the function $f(z) = z^4 - 5z + 1$

$i$) First we count the zeros in the set $\{|z| < 2\}$ :
Let $g_1(z) = z^4$. Then on $C_2 = \{|z| = 2\}, |f(z) - g_1(z)| = |-5z + 1| \le 5|z| + 1 = 11 < 16 = |g_1(z)|$. So the number of zeros of $f$ in $\{|z| < 2\}$ equals the number of zeros of $g_1$ in $\{|z| < 2\}$ which is 4 ($g_1$ has a zero of order 4 at 0).

$ii$) We now count the zeros in the set $\{|z| < 1\}$ :
Let $g_2 = -5z$. Then on $C_1 = \{|z| = 1\}, |f(z) - g_2(z)| = |z^4 + 1| \le |z|^4 + 1 = 2 < 5 = |g_2(z)|$. So the number of zeros of $f$ in $\{|z| < 1\}$ equals the number of zeros of $g_2$ in $\{|z| < 1\}$ which is 1 ($g_2$ has a zero of order 1 at 0).

Therefore $f$ has $4 - 1 = 3$ zeros on $\{|z| < 2\} \setminus \{|z| < 1\} = \{1 \le |z| < 2\}$.

Finally, when $|z| = 1, |f(z)| \ge |-5z| - |z|^4 - 1 = 5 - 2 = 3 > 0 \Rightarrow f(z) \ne 0$ when $|z| = 1$. So $f$ has 3 zeros on $\{1 < |z| < 2\}$.

□

7. Show there is exactly one point $z$ in the right half plane $\{z : Re(z) > 0\}$ at which $z + e^{-z} = 2$. Hint: Consider the countour in the right half plane enclosing the (nearly) half disk bounded by $\{|z| = R\}$ and the vertical line $Re(z) = \epsilon$, for $R > 3$ and $\epsilon > 0$ small (so in particular, the ball of radius 1 centered at 2 is contained inside the contour).

*Proof.* Let $\gamma$ be the contour in the hint. Then using the fact $|e^z| = e^{Rez}$, on $\gamma$ we have $|e^{-z}| = e^{-Rez} < e^0 = 1$, since $\gamma$ lies in $\{z : Re(z) > 0\}$ and $e^x$ is a monotonically increasing (real) function.

Now let $f(z) = z + e^{-z} - 2$ and $g(z) = z - 2$. On $\gamma$, $|f - g| = |e^{-z}| < 1 < |z - 2| = g(z)$, since the ball of radius 1 centered at 2 is contained inside $\gamma$, so all points on $\gamma$ are distance greater than 1 from 2.

So now applying Rouche's Theorem, we have the number of zeros of $f$ in $\gamma$ equals the number of zeros of $g$ in $\gamma$, which equals 1, since $z - 2$ has a zero at 2, which is inside $\gamma$.

Finally, this is the *only* solution in the right half plane, because we can let $R \to \infty, \epsilon \to 0$, and the previous argument remains valid (i.e. $f$ still only has one zero inside $\gamma$). □

8. Show if $p(z) = z^n + a_{n-1}z^{n-1} + ... + a_1 z + a_0$, then there must be at least one point $z_0$ with $|z_0| = 1$ such that $|p(z_0)| \ge 1$. Hint: If $|p(z)| < 1$ everywhere on $\{|z| = 1\}$, how many zeros must $q(z) = a_{n-1}z^{n-1} + ... + a_1 z + a_0$ have?

*Proof.* Suppose for contradiction that $|p(z)| < 1$ everywhere on $\{|z| = 1\}$. Let $g(z) = -z^n$, $f(z) = p(z) + g(z) = a_{n-1}z^{n-1} + ... + a_1 z + a_0 = q(z)$. Then on $\{|z| = 1\}$,

$$|f(z) - g(z)| = |p(z)| < 1 = 1^n = |g(z)|.$$

So by Rouche's Theorem $f$ and $g$ have the same number of zeros in the unit disk. But this is a contradiction, because $g$ has $n$ zeros in the unit disk, but $f$ can have at most $n-1$ zeros there, being a polynomial of degree $n-1$. □

9. Let $f(z) = \frac{z-1}{z+1}$. What is the image under f of:
*a*. The real axis (Hint: this is a *line*)
*b*. $\{|z| = 2\}$ (Hint: this is a *circle*)
*c*. $\{|z| = 1\}$
*d*. The imaginary axis

*Proof.* *a*. Since $f$ is an LFT, $f$ sends lines to circles or lines. So it suffices to check where three distinct points of $\mathbb{R}$ are sent to find its image under $f$. Note $f(1) = 0, f(\infty) = 1$, and $f(-1) = \infty$. So $f$ sends the real axis to the line through 0 and 1, i.e. the real axis.

*b*. Again, $f$ is an LFT, so it sends circles to circles or lines. And $f(2) = \frac{1}{3}, f(-2) = 3,$ and $f(2i) = \frac{-1+2i}{1+2i} = \frac{1}{5}(-1+2i)(1-2i) = \frac{1}{5}(3+4i)$. Since these points don't lie on a line, $f$ maps $\{|z| = 2\}$ to the circle through these points.

*c*. Since $f(-1) = \infty$, the image of $\{|z| = 1\}$ is a line. To find which, we map two other points: $f(1) = 0, f(i) = \frac{i-1}{i+1} = \frac{(i-1)^2}{2} = -i$. Thus $f$ maps $\{|z| = 1\}$ to the line through 0 and $-i$, i.e. the imaginary axis.

*d*. You may recall $f(z) = \frac{z-1}{z+1}$ maps the right half plane to the unit disk, sending the boundary to the boundary. To veryify this, if $z = iy$, $|f(z)| = |\frac{iy-1}{iy+1} = |(-1)\frac{iy-1}{-iy-1}| = 1$, since $-iy-1$ is the conjugate of $iy-1$. Thus the image of the imaginary axis under $f$ is a subset of the unit circle. But since $f$ is invertible, and the imaginary axis is sent to a line or a circle, it follows that the $f$ sends the imaginary axis to the entire unit circle.

□

10. Suppose $a, b, c, d \in \mathbb{R}, ad > bc$. Show $T(z) = \frac{az+b}{cz+d}$ leaves the upper half plane $\mathbb{H}$ invariant (i.e. $T$ sends the upper half plane to itself).

*Proof.* First we note that since $T$ is an LFT, it sends lines to circles or lines. $T$ clearly then sends the real line to itself, since if $x \in \mathbb{R}$, both the numerator and denominator of $T(x) = \frac{ax+b}{cx+d}$ are real numbers (if $x = \frac{-d}{c}$, then $T(x) = \infty$, which still lines on the real line).

Therefore by the continuity of $T$, it suffices to check that $T(z_0)$ lands in the upper half plane, for any $z_0 \in \mathbb{H}$. So let $z_0 = i$. Then $T(i) = \frac{ai+b}{ci+d} = \frac{(b+ai)(d-ci)}{c^2+d^2} = \frac{(bd+ac)+i(ad-bc)}{c^2+d^2}$. So then $Im(T(i)) = \frac{(ad-bc)}{c^2+d^2} > 0$, since $ad - bc > 0$, by assumption. Thus $T(i) \in \mathbb{H}$, so we're done. □

11. Find a bijective conformal map that takes a bounded region of $\mathbb{C}$ to an unbounded region.

*Proof.* Take $f(z) = \frac{1}{z}$. Then $f$ sends the bounded set $\{0 < |z| < 1\}$ to the unbounded set $\{|z| > 1\}$. Moreover $f$ is conformal on $\{0 < |z| < 1\}$, since $f$ is analytic there, and $f'(z) = \frac{-1}{z^2} \neq 0, \ \forall z \in \mathbb{C} \setminus \{0\}$, $\square$

12. What is the image of the region $A = \{x + iy : xy > 1, x > 0, y > 0\}$ under the transformation $f(z) = z^2$?

*Proof.* Note, in the right half plane $\{Re(z) > 0\}, f(z) = z^2$ is conformal, since $f'(z) = 2z = 0$ iff $z = 0$. So it suffices to map the boundary of $A$ and see where a point in the interior is sent. The boundary of $A$ is the part of the curve $xy = 1$ in the first quadrant $\{x > 0, y > 0\}$. Now $f(x+iy) = x^2 - y^2 + i2xy$, so the image of $xy = 1$ is the set $v = 2$ in the $u, v$ plane. So now the interior point $2 + 2i = 2\sqrt{2}e^{\frac{i\pi}{4}}$ is sent to $8i$ under $f$. So the image of $A$ under $f$ is the set $\{v > 2\}$. $\square$

13. Let $A$ be the upper half of the unit disk $\{|z| < 1\}$. Find the temperature $T$ inside $A$ if the circular portion of the boundary is insulated, and $T = 0$ for $0 < x < 1$ on the real axis, and $T = 10$ for $-1 < x < 0$ on the real axis.

*Proof.* Use the map $\log(z)$ to make $A$ to the half strip $B = \{u < 0, 0 < v < \pi\}$ in the $u, v$ plane, where the temperature is 10 when $v = \pi$ and 0 when $v = 0$. Then the in the half-strip, we have $T_0 = \frac{10v}{\pi}$. Therefore the tempterature in the $x, y$ plane is given by $T(x, y) = T_0(\log(x + iy)) = \frac{10}{\pi} tan^{-1}(\frac{y}{x})$. $\square$

14. Consider the region the entire unit disk $\{|z| < 1\}$. The electric potential is maintained at $\phi = 0$ on the lower semicicle and at $\phi = 1$ on the upper semicircle. Find the value of $\phi$ inside.

*Proof.* We apply our general procedure for the Dirichlet problem by mapping the unit disk to the upper half plane. We can do this with the LFT

$$u + iv = f(x + iy) = f(z) = \frac{1}{i}\frac{z + 1}{z - 1} = \frac{1}{i}\frac{(x + 1) + iy}{(x - 1) + iy}$$

So then $u = \frac{-2y}{(x-1)^2+y^2}$ and $v = \frac{1-x^2-y^2}{(x-1)^2+y^2}$. We can use the standard solution on the upper half plane:

$$\phi_0(u, v) = 0 + \frac{1}{\pi}(1 - 0)\tan^{-1}\frac{v}{u} = \frac{1}{\pi}\tan^{-1}\frac{v}{u}$$

The solution on the unit disk is then

$$\phi(x, y) = \phi_0(f(x, y)) = \phi_0(u, v) = \frac{1}{\pi}\tan^{-1}\Big(\frac{x^2 + y^2 - 1}{2}\Big),$$

where the values of arctangent must be taken between 0 and $\pi$. $\square$

15. Find the flow around the upper half of the unit circle if the velocity is parallel to the $x$ axis and is $\alpha$ at $\infty$. (Here $A$ is the region of the upper-half plane exterior to the unit circle, i.e. $A = \{\text{z} : Im(z) > 0, |z| > 1\}$.

*Proof.* Use the mapping $z \mapsto z + \frac{1}{z}$. This maps $A$ to the upper half plane, and $F_0(z) = \alpha z$ is the complex potential in the upper half plane, so the potential in $A$ is given by $F(z) = \alpha(z + \frac{1}{z})$. And then $\phi(r,\theta) = \alpha\left(r + \frac{1}{r}\right)\cos\theta, \psi(r,\theta) = \alpha\left(r - \frac{1}{r}\right)\sin\theta$. $\square$

16. Let $f$ be analytic in a *domain* (open and connected) $A$, and let $z_1, z_2 \in A$. Let $f'(z_1) \neq 0$. Show $f$ is not constant on a neighborhood of $z_2$.

*Proof.* Suppose for contraction $f = c$ constant, on a neighborhood $B(z_2, r)$ of $z_2$. Then since $B(z_2, r) \subset A$ (connected) has accumulation points (every point is one in fact), the Identity Principle implies that $f = c$ constant on $A$, since $\{z \in A : f(z) - c = 0\}$. has an accumulation point in $A$. But if that were the case, then $f'(z) = 0\ \forall z \in A$, so in particular, $f'(z_1) = 0$, a contraction. Therefore $f$ is non-constant on all neighborhoods of $z_2$. $\square$