Topology Qual Workshop Day 1: Compactness.

Warm-up Problems:

- If X is T_2 and $A \subseteq X$ is compact, then A is closed.
- If X is compact, and $A \subseteq Y$ is closed, then A is compact.
- (June '11 # A1) If $f : X \to Y$ is continuous and $A \subseteq X$ is compact, then $f(A)$ is compact.

(1) (Jan '02 # A1) Let X be a compact space, and let

$$A_1 \supseteq A_2 \supseteq \cdots \supseteq A_n \supseteq \cdots$$

be a descending chain of non-empty closed subsets of X. Show that their intersection $\cap_{n=1}^{\infty} A_n$ is non-empty.

(2) (June '04 # A1 [and June '07 # A4]) A topological space X is *locally compact* if every point in X has an open neighborhood whose closure is compact. Show that the Cartesian product of two locally compact spaces, with the product topology, is also locally compact.

(3) (June '05 # A2) Let X be the set of integers, let $\mathcal{C}_1 := \{A \subseteq X | X \setminus A \text{ is finite }\}$, and let $\mathcal{C}_2 := \{A \subseteq X | 0 \notin A\}$. Show that the union $\mathcal{T} := \mathcal{C}_1 \cup \mathcal{C}_2$ is a topology on X, and show that this topological space is compact.

(4) (Jan '06 # A4) Let $\mathcal{T}, \mathcal{T}'$ be two topologies on X. Show that if $(X, \mathcal{T})$ is compact and Hausdorff, $\mathcal{T} \subseteq \mathcal{T}'$, and $\mathcal{T} \neq \mathcal{T}'$, then $(X, \mathcal{T}')$ is Hausdorff but *not* compact.

(5) (June '05 # A4) A topological space X is called *metacompact* if for every open cover C of X, there is a subcover $C\acute{}$ satisfying the property that for every point $p \in X$, there are only finintely many open sets in $C\acute{}$ containing p. w
  (a) Show that metacompactness is a homeomorphism invariant.
  (b) Let X be the integers with the topology $\tau = \{U \subset X : 0 \in U\} \cup \{\emptyset\}$. Show that this space is not metacompact.

(6) (June '11 # A4) Suppose A, B are disjoint, compact subspaces of the Hausdorff topological space X. Prove that there are open subsets U, V of X such that $A \subseteq U, B \subseteq V$ and $U \cap V = \emptyset$.

(7) Suppose that X is compact and Y is Hausdorff. Prove that every one-to-one, onto, continuous map $f : X \to Y$ is a homeomorphism.

(8) (Purdue '11) Let X be a set with two elements $\{a, b\}$. Give X the indiscrete topology. Give $X \times \mathbb{R}$ the product topology. Let $A \subset X \times \mathbb{R}$ be $(\{a\} \times [0,1]) \cup (\{b\} \times (0,1))$. Prove that A is compact.

(9) (Purdue '11) Let X be a compact space and let $\{C_\alpha\}_{\alpha \in A}$ be a collection of closed sets in X. Let $C = \cap_{\alpha \in A} C_\alpha$ and let U be an open set containing C. Prove there is a finite set $\alpha_1, ...\alpha_n$ in A with $C_{\alpha_1} \cap \cdots \cap C_{\alpha_n} \subset U$.