Topology Qual Workshop Day 2: Connectedness and Path Connectedness

Warm-up Problems:

- If X is a connected space if and only if X and the empty set are the only open and closed sets.
- If X is a path connected space, then X is connected.
- (Purdue '10) Prove or disprove: If X is path connected, and $f : X \to Y$ is continuous, then $f(x)$ is path connected.

(1) (June '08 # A2) Let $p : X \to Y$ be a quotient map. Show that if Y is connected and moreover each set $p^{-1}(\{y\})$ is connected, then X is connected.

(2) (Jan '08 # B7) Show that if A is a proper subset of a connected space X and B is a proper subset of a connected space Y, then $(X \times Y) \setminus (A \times B)$ is connected.

(3) (June '07 # A2) A space $(X, \mathcal{T})$ is called *locally path-connected* if, for every $x \in X$, every neighborhood of x contains a path-connected neighborhood of x. Show that a connected, locally path-connected space is path-connected.

(4) (Jan '06 # A5) Recall that a topological space X is *locally connected* if for every point $x \in X$ and every neighborhood U of x there exists a connected neighborhood V of x with $V \subseteq U$.

(a) Prove that a topological space X is localy connected iff for every open set $U \subseteq X$ the components of U are open.

(b) Now let $p : X \to Y$ be a quotient map. Prove that if C is a component of an open subset $U \subset Y$ then $p^{-1}(C)$ is a union of components of $p^{-1}(U)$.

(c) Deduce that if X is locally connected, then so is Y.

(5) (June '05 # A3) Let X be a connected space with connected subset Y, and suppose that $X \setminus Y$ is not connected, with $X \setminus Y = A \cup B$ a separation of $X \setminus Y$. Show that if B is open in X, then $A \cup Y$ is a connected subset of X.

(6) (Jan '02 # A3) Let X be a topological space, and $A, B \subset X$ be connected subsets of X. Show that if $A \cap \overline{B} \neq \emptyset$ then $A \cup B$ is a connected subset of X.

(7) (Jan '02 # A5) Show that $\mathbb{R}$ and $\mathbb{R}^2$ (with their usual topologies) are not homeomorphic.

(8) (June '10 # A4) Let $Z = X \cup Y$, for X and Y connected subspaces of Z with $X \cap Y = \emptyset$. Let $x_0 \in X$ and $y_0 \in Y$. Let $\sim$ be the equivalence relation generated by the equivalence $x_0 \sim y_0$. Show that the quotient space $Z/\sim$ is connected.