Day 2: Metric spaces and topology

Definition. A set E is *countable* if a bijective function $f : E \to \mathbb{N}$ exists.

Proposition 1.1.

i) If a surjective function $g : E \to F$ exists and E is countable, then F is countable.

ii) If an injective function $h : E \to F$ exists and F is countable, then E is countable.

iii) If each E_j is countable for $j = 1, 2, \ldots$, then $\bigcup_{j=1}^{\infty} E_j$ is countable.

Definition. A *metric space* is a pair (X, d) consisting of a set X along with a function $d : X \times X \to \mathbb{R}$ such that for any $p, q \in X$,

i) $d(p, q) > 0$ if $p \neq q$; $d(p, p) = 0$

ii) $d(p, q) = d(q, p)$

iii) $d(p, q) \leq d(p, r) + d(r, q)$ for any $r \in X$.

The metric d gives rise to a topology on X with basis given by the neighborhoods

$$N_r(p) = \{x \in X : d(x, p) < r\}.$$

A set $G \subset X$ is open this topology if every point in G is an interior point. A metric space is called separable if it contains a countable dense subset.

Definition. A *normed space* $(X, ||\cdot||)$ is a vector space X over a field $\mathbb{F}$ (nearly always $\mathbb{R}$ or $\mathbb{C}$) along with a norm $||\cdot|| : X \to \mathbb{R}$ such that for all $x, y \in X$,

i) $||x|| > 0$ if $x \neq 0$, $||0|| = 0$.

ii) $||\alpha x|| = |\alpha| ||x||$ for $\alpha \in \mathbb{F}$.

iii) $||x + y|| \leq ||x|| + ||y||$.

Every normed space $(X, ||\cdot||)$ is a metric space by taking $d(x, y) = ||x - y||$. The converse is not true in general—even if X is a vector space.

Definition. An *inner product space* is a vector space X over a field $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$ along with an inner product $\langle \cdot, \cdot \rangle : X \times X \to \mathbb{F}$ such that for all $x, y, z \in X$,

i) $\langle x, y \rangle = \overline{\langle y, x \rangle}$.

ii) $\langle \alpha x + y, z \rangle = \alpha \langle x, z \rangle + \langle y, z \rangle$.

iii) $\langle x, x \rangle \geq 0$, and if $\langle x, x \rangle = 0$ then $x = 0$.

Every inner product space is a normed space via the induced norm $||x|| := \sqrt{\langle x, x \rangle}$.

Theorem 1.2 (Heine-Borel / Rud76, Thm. 2.41)**.** *If $E \subset \mathbb{R}^n$ has any one of these properties, then it has the other two:*

i) E is closed and bounded.

ii) E is compact.

iii) Every infinite subset of E has a limit point in E.

Note that the assumption that E is a subset of $\mathbb{R}^n$ (with the usual topology) is *essential*.

Theorem 1.3 (Weierstrass / Rud76, Thm. 2.42)**.** *Every bounded infinite subset of* $\mathbb{R}^n$ *has a limit point in* $\mathbb{R}^n$*.*

Proposition 1.4. *Every open set* $G \subset \mathbb{R}$ *can be written as a finite or countable union of disjoint open intervals* (a_j, b_j) *with at most one* $a_j = -\infty$ *and at most one* $b_j = \infty$*.*

Definition. A metric space (X, d) is *connected* if whenever $U, V \subseteq X$ are open and satisfy $U \cap V = \emptyset$ and $U \cup V = X$, then either $U = \emptyset$ or $V = \emptyset$.

Warm-up problems. Throughout, X is a metric space.

1) State the definition of what it means for a set $K \subset X$ to be compact, including the definition of open cover.
2) (June 2010 #2b) Prove that $\{1/n : n \in \mathbb{Z} \setminus \{0\}\} \cup \{0\}$ is compact using the above definition.
3) Is the set of all sequences $x_1, x_2, \ldots$ with $x_i \in \{0, 1\}$ for $i = 1, 2, \ldots$ countable?
4) Show that an inner product space satisfies the parallelogram law with its induced norm:
$$||x + y||^2 + ||x - y||^2 = 2||x||^2 + 2||y||^2.$$
(For simplicity, you may assume that this is an inner product space over $\mathbb{R}$.)
5) (January 2010 #1a partial) Determine whether or not the sets $\{(x, y) \in \mathbb{R}^2 : x + y < 3\}$ and $\{f \in C([-1, 1]) : f(0) = 0\}$ are open, closed, or compact, where $C([-1, 1])$ is considered with $\|\cdot\|_\infty$.

Problems.

6) (May 2019, #1) Let (M, d_M), (N, d_N) be metric spaces. Define $d_{M\times N} : (M\times N)\times(M\times N) \to \mathbb{R}$ by
$$d_{M\times N}((x_1, y_1), (x_2, y_2)) := d_M(x_1, x_2) + d_N(y_1, y_2).$$
   1) Prove that $(M \times N, d_{M\times N})$ is a metric space.
   2) Let $S \subseteq M$ and $T \subseteq N$ be compact sets in (M, d_M) and (N, d_N), respectively. Prove that $S \times T$ is a compact set in $(M \times N, d_{M\times n})$.
7) (June 2003, #1b,c) (b) Show by example that the union of infinitely many compact subsets of a metric space need not be compact. (c) If (X, d) is a metric space and $K \subset X$ is compact, define $d(x_0, K) = \inf_{y\in K} d(x_0, y)$. Prove that there exists a point $y_0 \in K$ such that $d(x_0, K) = d(x_0, y_0)$.
8) (January 2009, #4a) Consider the metric space $(\mathbb{Q}, d)$ where $\mathbb{Q}$ denotes the rational numbers and $d(x, y) = |x - y|$. Let $E = \{x \in \mathbb{Q} : x > 0,\ 2 < x^2 < 3\}$. Is E closed and bounded in $\mathbb{Q}$? Is E compact in $\mathbb{Q}$?
9) (January 2011 #3a) Let (X, d) be a metric space, $K \subset X$ be compact, and $F \subset X$ be closed. If $K \cap F = \emptyset$, prove that there exists an $\epsilon > 0$ so that $d(k, f) \geq \epsilon$ for all $k \in K$ and $f \in F$.
10) Let (X, d) be an unbounded and connected metric space. Prove that for each $x_0 \in X$, the set $\{x \in X :\ d(x, x_0) = r\}$ is nonempty.

Additional Problems.

11) Show that if $f : \mathbb{R} \to \mathbb{R}$ is monotone increasing, then f has at most a countable set of jump discontinuities.
12) Prove Proposition 1.4.
13) Verify the remark following the definition of normed space.