Topology Qual Workshop Day 3: Separation Axioms

Warm-up Problems:

- If (X, d) is a metric space, then it is Hausdorff [see number 8].
- $T_4 \implies T_3 \implies T_2 \implies T_1 \implies T_0$
- Suppose (X, τ) is a topological space and let (X, τ') be same set with the finite complement topology. Then X is T_1 if and only if $\text{id} : (X, \tau) \to (X, \tau')$ is continuous.

(1) (Jan '08 # B9) Let $X = \mathbb{R}^2$ and define an equivalence relation on X by $(x_1, x_2) \sim (y_1, y_2)$ iff they are qual or $x_1 = y_1 = 0$. Set $Y = X/\sim$. Show that Y is Hausdorff but does not have a countable basis for its topology.

(2) (June '06 # A5)
   (a) Recall that a topological space is called *separable* if it contains a countable dense subset. Prove that a countable product of separable spaces (with the product topology is separable).
   (b) Prove that the result in the first part is false if the product is given the box topology.

(3) (June '04 # A3) Show that for a topological space X, if every $x \in X$ has a neighborhood whose closure is a regular space, then X is regular.

(4) (Jan '04 # A2) Show that if the metric space (X, d) is separable, then the metric topology on X is second countable.

(5) (Jan '04 # A4) Two subsets $A, B \subseteq X$ of the space (X, τ) are called *separated* if there are $U, V \in \tau$ with $A \subseteq U \subseteq X \setminus B$ and $B \subseteq V \subseteq X \setminus A$. We say that X is *completely normal* if X is T_1 and if for every pair of separated subsets A, B there are $U, V \in \tau$ so that $A \subseteq U, B \subseteq V$ and $U \cap V = \emptyset$. Show that a space (X, τ) is completely normal $\Leftrightarrow$ every subset of X is normal.

(6) (Jan '12 # A2) Let X and Y be two topological spaces and let $X \times Y$ be endowed with the product topology. Prove that if X and Y each have a countable dense subset, then so does $X \times Y$.

(7) (Jan '12 # A4)
   (a) State the definition of a what it means for a topological space to be regular.
   (b) Prove that a subspace of a regular space is also regular.
   (c) Prove that a product of two regular spaces (equipped with the product topology) is also regular.

(8) (June '11 # A3) Prove that every metrizable space is normal Hausdorff (aka T_4).

(9) (June '09 # A4) Let (X, τ) and (X, τ') be topological spaces with $\tau \subseteq \tau'$.
   (a) If (X, τ') is normal, must (X, τ) also be normal?
   (b) If (X, τ') is compact, must (X, τ) also be compact?