Day 3: Sequences and series

Relevant information. A metric space (X, d) is called complete if every Cauchy sequence in X converges in X. For a real-valued sequence $\{a_k\}$ the limit superior and inferior are given by

$$\limsup_{n\to\infty} a_n = \lim_{n\to\infty} \sup\{a_k : k \geq n\}$$
$$\liminf_{n\to\infty} a_n = \lim_{n\to\infty} \inf\{a_k : k \geq n\}$$

Note that $\{a_n\}$ converges to $L \in \mathbb{R}$ if and only if $\limsup a_n = \liminf a_n = L$.

Theorem 2.1 (c.f. [Rud76, Thm. 3.14]). *A monotone increasing (resp., decreasing) sequence converges if and only if it is bounded above (resp., below).*

Theorem 2.2 ("nth term test" / [Rud76, Thm. 3.23]). *If $\sum_{k=1}^{\infty} a_k$ converges, then $\lim_{k\to\infty} a_k = 0$.*

Theorem 2.3 (Cauchy condensation test / [Rud76, Thm. 3.27]). *If $a_1 \geq a_2 \geq \cdots \geq 0$ then $\sum_{n=1}^{\infty} a_n$ converges if and only if $\sum_{k=1}^{\infty} 2^k a_{2^k}$ converges.*

Theorem 2.4 (Root test / [Rud76, Thm. 3.33]). *Set $\alpha = \limsup_{n\to\infty} \sqrt[n]{|a_n|}$. Then,*

i) If $\alpha < 1$, then $\sum a_n$ converges;

ii) If $\alpha > 1$, then $\sum a_n$ diverges;

iii) If $\alpha = 1$, the test gives no information.

Theorem 2.5 (Ratio test / [Rud76, Thm. 3.34]). *The series $\sum a_n$*

- *converges if $\limsup_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| < 1$;*
- *diverges if $\left|\frac{a_{n+1}}{a_n}\right| \geq 1$ for all $n \geq n_0$.*

Theorem 2.6 ([Rud76, Thm. 3.39]). *Given the power series $\sum c_n z^n$, put*

$$\alpha = \limsup_{n\to\infty} \sqrt[n]{|c_n|}, \qquad R = \frac{1}{\alpha}.$$

Then $\sum c_n z^n$ converges if $|z| < R$ and diverges if $|z| > R$. R is called the radius of convergence of $\sum c_n z^n$.

Warm-up problems.

1) For a real sequence $\{x_n\}$, if $\lim_{n\to\infty} x_n = x$ and $\lim_{n\to\infty} x_n = y$ then $x = y$.

2) If X is a metric space, $E \subset X$, and x is a limit point of E, then there exists a sequence $\{x_n\} \subset E$ which converges to x.

3) (January 2003 #1) Let $\{a_k\}$ be a sequence of real numbers such that the series $\sum_{k=1}^{\infty} a_k$ converges and $\sum_{k=1}^{\infty} a_k^2$ diverges. Prove that $\sum_{k=1}^{\infty} a_k$ does not converge absolutely.
(See also June 2010 #3a where you are instead told that $\sum_{k=1}^{\infty} a_k a_{k+1}$ diverges and asked to show the same result. Compare this to June 2009 #3a and January 2005 #1b.)

4) ([KRD10, #3.1.D]) Let $\{a_n\}$ be a sequence such that $\lim_{n\to\infty} |a_n| = 0$. Prove that there is a subsequence of $\{a_{n_k}\}$ of $\{a_n\}$ such that $\sum_{k=1}^{\infty} a_{n_k}$ converges.

5) (c.f. [Abb01, Exercise 2.4.5]) Let $x_1 = 2$ and define

$$x_{n+1} = \frac{1}{2}\left(x_n + \frac{2}{x_n}\right).$$

Find $\lim_{n\to\infty} x_n$. *Hint:* Show that $\{x_n\}$ is decreasing.

Problems.

6) (June 2013 #1a) Let $a_n = \sqrt{n}\left(\sqrt{n+1} - \sqrt{n}\right)$. Prove that $\lim_{n\to\infty} a_n = 1/2$.

7) (January 2014 #2) (a) Produce sequences $\{a_n\}$, $\{b_n\}$ of positive real numbers such that

$$\liminf_{n\to\infty}(a_n b_n) > \left(\liminf_{n\to\infty} a_n\right)\left(\liminf_{n\to\infty} b_n\right).$$

(b) If $\{a_n\}$, $\{b_n\}$ are sequences of positive real numbers and $\{a_n\}$ converges, prove that

$$\liminf_{n\to\infty}(a_n b_n) = \left(\lim_{n\to\infty} a_n\right)\left(\liminf_{n\to\infty} b_n\right).$$

8) (May 2011 #4a) Determine the values of $x \in \mathbb{R}$ for which $\displaystyle\sum_{n=1}^{\infty} \frac{x^n}{1+n|x|^n}$ converges, justifying your answer carefully.

9) (June 2005 #3b) If the series $\sum_{n=0}^{\infty} a_n$ converges conditionally, show that the radius of convergence of the power series $\sum_{n=0}^{\infty} a_n x^n$ is 1.

10) (January 2011 #5) Suppose $\{a_n\}$ is a sequence of positive real numbers such that $\lim_{n\to\infty} a_n = 0$ and $\sum a_n$ diverges. Prove that for all $x > 0$ there exist integers $n(1) < n(2) < \ldots$ such that $\sum_{k=1}^{\infty} a_{n(k)} = x$.
(Note: Many variations on this problem are possible including more general rearrangements. You may also wish to show that if $\sum a_n$ converges conditionally then given any $x \in \mathbb{R}$ there is a rearrangement of $\{b_n\}$ of $\{a_n\}$ such that $\sum b_n = r$. See Rudin Thm. 3.54 for a further generalization.)

11) (June 2008 # 4b) Assume $\beta > 0$, $a_n > 0$, $n = 1, 2, \ldots$, and the series $\sum a_n$ is divergent. Show that $\displaystyle\sum \frac{a_n}{\beta + a_n}$ is also divergent.

More Problems.

12) (January 2012 #1a) Let $\{a_n\}$, $\{b_n\}$ be bounded sequences of positive real numbers. If $\sum b_n$ is convergent, show that $\sum a_n b_n$ is also convergent.

13) Assume that Theorem 2.4 (the root test) is true and prove the ratio test (Theorem 2.5).

14) (January 2008 #6b) Suppose that $\lim_{n\to\infty} s_n = s$ and $\lim_{n\to\infty} t_n = t$ with $s \neq t$ and $s_n \neq t_n$ for all n. Use and ϵ-δ proof to show that

$$\lim_{n\to\infty} \frac{s_n + t_n}{s_n - t_n} = \frac{s+t}{s-t}.$$

15) Prove theorems 2.1, 2.2, and 2.3.

16) (January 2006 #5) Let $a_{m,n} \geq 0$ for $m, n \in \mathbb{N}$ and assume that the partial sums

$$\sum_{m=1}^{M}\sum_{n=1}^{N} a_{m,n}$$

are bounded above. Prove carefully that $\sum_{m=1}^{\infty}\left(\sum_{n=1}^{\infty} a_{m,n}\right)$ and $\sum_{n=1}^{\infty}\left(\sum_{m=1}^{\infty} a_{m,n}\right)$ exist and are equal.

References

[Abb01] Stephen Abbott. *Understanding Analysis*. Springer, 2001.
[KRD10] Allan P. Donsig Kenneth R. Davidson. *Real analysis and applications*. Springer, 2010.
[Rud76] Walter Rudin. *Principles of mathematical analysis*. McGraw-Hill, Inc., USA, third edition, 1976.