Topology Qual Workshop Day 7: Fundamental Groups

Warm-up Problems:

- Let $G = \langle x, y | x^2yxy = 1 \rangle$. Describe a space X that has $\pi_1(X) = G$.
- Find two spaces X and Y such that $\pi_1(X) \cong \pi_1(Y)$ and are non trivial, and are NOT homeomorphic.
- Find $\pi_1(T^2)$, where $T^2 = S^1 \times S^1$ is the standard two torus, in TWO different ways.

(1) (Michigan May '09) Let X be the space obtained from $S^1 \times \mathbb{R}$ by removing the interior of k disjoint 2-disks.

(a) Compute the fundamental group $\pi_1(X)$.

(b) What would be your answer to part (a) if $S^1 \times \mathbb{R}$ is replaced by $S^2 \times \mathbb{R}$ and 2-disks are replaced by 3-balls?

(c) Let Y be the union of two copies of the real projective plane $\mathbb{R}P^2$ having exactly one point y in common. Compute $\pi_1(Y, y)$.

(2) State the Seifert-van Kampen Theorem and use it to show that $\pi_1(S^n) = 0$ for $n > 1$. Why doesn't this argument work for $n = 1$?

(3) (Wisconsin Aug '10) The graph K has six vertices $a_1, a_2, a_3, b_1, b_2, b_3$ and nine edges a_ib_j for $i, j = 1, 2, 3$. The space X obtained from K by attaching a 2-cell along each loop formed by a cycle of four edges in K. Find $\pi_1(X)$.

(4) (Jan '02) Find the fundamental group of the space X consisting of $\mathbb{R}^3$ with the three coordinate axes removed.

(5) (June '05) Find a cell structure for the quotient space obtained by identifying two distinct points a, b in a 2-torus to a third point c in a 2-sphere, and compute a presentation for the fundamental group of this space.

(6) (June '08) Let X be the triangle parachute formed from the standard 2-simplex Δ^2 by identifying the three vertices with one another. Compute a presentation for $\pi_1(X)$ and show that $\pi_1(X)$ is isomorphic to a free group F_n (and identify which n!).

(7) Attach a 2-disc D^2 to a torus $S^1 \times S^1$ by the attaching map $e^{2\pi it} \mapsto \left(e^{2\pi it}, e^{2\pi it}\right)$, thinking of the boundary ∂D^2 and each S^1 factor as the unit circle in the complex plane, and let X be the resulting space. Compute a presentation for the fundamental group of X.

(8) (June '11) Let X be the space obtained by deleting three distinct points from $\mathbb{R}^2$. Compute $\pi_1(X)$.

(9) (Jan '12) Let $X = \mathbb{R}^3 \backslash S^1$, the complement of a single circle in $\mathbb{R}^3$.

(a) Describe a deformation retraction of X onto $S^1 \vee S^2$.

(b) Compute $\pi_1(X)$.

(10) (June '12) Let X be the space obtained from the torus $T^2 = S^1 \times S^1$ by gluing in a disk D^2 along its boundary $\partial D^2 = S^1$ using the map $\alpha : S^1 \to T^2$ given by $z \mapsto (z, (1, 0))$ for $z \in S^1$.

(a) Find $\pi_1(X)$.

(b) Give a CW complex for X.

Covering Space Problems:

- (June '10) Use covering space theory to show that if H is a subgroup of index 3 in a finitely presented group G, then H is finitely presented.

- Let $F(a, b)$ be the free group of two generators. What is the presentation complex and the Cayley complex for $F(a, b)$? Find an index 4 subgroup and the covering space that corresponds to it.