Day 7: Sequences and series of functions

Relevant information.

Definition. For a sequence of functions $\{f_n\}$ where $f_n, f : E \to \mathbb{R}$ for all n,

i) $f_n \to f$ pointwise if $\lim_{n\to\infty} f_n(x) = f(x)$ for each $x \in E$,

ii) $f_n \to f$ uniformly if for every $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that $|f_n(x) - f(x)| < \epsilon$ for all $n \geq N$ and all $x \in E$. That is, $f_n \to f$ uniformly provided $||f_n - f||_\infty \to 0$.

iii) $\sum_{n=1}^\infty f_n(x) \to f(x)$ provided the partial sums $\sum_{n=1}^N f_n(x)$ converge pointwise to f as $N \to \infty$.

iv) $\sum_{n=1}^\infty f_n(x) \to f(x)$ provided the partial sums $\sum_{n=1}^N f_n(x)$ converge uniformly to f as $N \to \infty$.

Theorem 6.1 ([Rud76, Thm. 7.12]). *If $\{f_n\}$ is a sequence of continuous functions on E and $f_n \to f$ uniformly on E, then f is continuous on E.*

Theorem 6.2 (Weierstrass M-test / [Rud76, Thm. 7.10]). *Suppose $\{f_n\}$ is a sequence of functions on E and that there exists a real sequence $\{M_n\}$ such that $|f_n(x)| \leq M_n$ for all $x \in E$. If $\sum_{n=1}^\infty M_n$ converges, then $\sum_{n=1}^\infty f_n$ converges uniformly on E.*

Theorem 6.3 ([Rud76, Thm. 7.16]). *If α is monotonically increasing on $[a, b]$, $f_n \in \mathscr{R}(\alpha)$ for all n, and $f_n \to f$ uniformly on $[a, b]$, then $f \in \mathscr{R}(\alpha)$, $\lim_{n\to\infty} \int_a^b f_n \, d\alpha$ exists, and*

$$\lim_{n\to\infty} \int_a^b f_n \, d\alpha = \int_a^b f \, d\alpha.$$

Theorem 6.4 ([Rud76, Thm. 7.17]). *Let $\{f_n\}$ be a sequence of functions differentiable on $[a, b]$ for which $\{f_n(x_0)\}$ converges at some $x_0 \in [a, b]$. If $\{f_n'\}$ converges uniformly on $[a, b]$ then $\{f_n\}$ converges uniformly on $[a, b]$ to a function f such that*

$$\lim_{n\to\infty} f_n'(x) = f'(x).$$

Theorem 6.5 (Arzelà-Ascoli / [KRD10, Thm. 8.6.9]). *Let $K \subset \mathbb{R}^n$ be compact. A collection of functions $\mathscr{F} \subset C(K, \mathbb{R}^m)$ is compact if and only if $\mathscr{F}$ is closed, bounded, and (pointwise) equicontinuous.*

Remark 6.6. Compare this to [Rud76, Thm. 7.25]. There are two common definitions of "equicontinuous." Rudin's defintion in 7.22 is sometimes called uniformly equicontinuous as δ does not depend on x or y.

Theorem 6.7 (Stone-Weierstrass / [Rud76, Thms. 7.26, 7.32]). *If f is a continuous function on $[a, b]$ then there exists a sequence of polynomials $\{P_n\}$ which converge uniformly to f.*
More generally: If $\mathscr{A}$ is a self-adjoint algebra of continuous functions on a compact set K which separates points of K and vanishes at no point of K, then given any $f \in C(K)$ there exists a sequence $f_n \subset \mathscr{A}$ such that $f_n \to f$ uniformly on K.

Warm-up problems.

1) Give a precise statement of the Stone-Weierstrass theorem for real-valued continuous functions. Then, verify that the set of all polynomials of the form

$$\left\{ \sum_{j=2017}^N a_j x^j : N \in \mathbb{N},\, N \geq 2017,\, a_j \in \mathbb{R} \right\}$$

along with the zero function is an algebra over $[-2,2] \subset \mathbb{R}$.

2) To clarify Remark 6.6: A family of functions $\mathscr{F} \subset C(K, \mathbb{R}^m)$ mapping a set $K \subset \mathbb{R}^n$ into $\mathbb{R}^m$ is pointwise equicontinuous on K provided for every $x \in K$ and $\epsilon > 0$ there exists some $\delta > 0$ (which may depend on x) such that $||f(x) - f(y)|| < \epsilon$ for all $f \in \mathscr{F}$ and $y \in K$ with $||x - y|| < \delta$. The family $\mathscr{F}$ is uniformly equicontinuous if for every $\epsilon > 0$ there exists some $\delta > 0$ such that $||f(x) - f(y)|| < \epsilon$ for all $f \in \mathscr{F}$ and $x, y \in K$ with $||x - y|| < \delta$. Prove that these definitions are equivalent when K is compact.
3) Assume that $\{f_n\}$ is a sequence of continuous functions $f_n : E \subset \mathbb{R} \to \mathbb{R}$ which converges uniformly to f. Prove the results of Theorem 6.1 directly.
4) Find the pointwise limit f of the sequence of functions $\{f_n\}$ given by $f_n(x) = x^n$ on $[0,1]$. Is the convergence of f_n to f uniform? ([KRD10, 8.6.A]) Why is $B = \{f \in C([0,1]) : ||f||_\infty \leq 1\}$ not compact?
5) Show that if $\{f_n\}$ is an equicontinuous sequence of functions on a compact set K and $f_n \to f$ pointwise on K, then $f_n \to f$ uniformly on K.

Problems.

6) (June 2010 #6a) Let $f : [0,1] \to \mathbb{R}$ be continuous with $f(0) \neq f(1)$ and define $f_n(x) = f(x^n)$. Prove that f_n does not converge uniformly on $[0,1]$.
7) (January 2008 5a) Let $f_n(x) = \frac{x}{1+nx^2}$ for $n \in \mathbb{N}$. Let $\mathcal{F} := \{f_n : n = 1, 2, 3, \ldots\}$ and $[a,b]$ be any compact subset of $\mathbb{R}$. Is $\mathcal{F}$ equicontinuous? Justify your answer.
8) (January 2005 #4, June 2010 #6b) If $f : [0,1] \to \mathbb{R}$ is continuous, prove that
$$\lim_{n\to\infty} \int_0^1 f(x^n)\, dx = f(0).$$
9) (January 2020 4a) Let $M < \infty$ and $\mathcal{F} \subseteq C[a,b]$. Assume that each $f \in \mathcal{F}$ is differentiable on (a,b) and satisfies $|f(a)| \leq M$ and $|f'(x)| \leq M$ for all $x \in (a,b)$. Prove that $\mathcal{F}$ is equicontinuous on $[a,b]$.
10) (June 2005 #5) Suppose that $f \in C([0,1])$ and that $\displaystyle\int_0^1 f(x)x^n\, dx = 0$ for all $n = 99, 100, 101, \ldots$. Show that $f \equiv 0$.
Note: Many variations on this problem exist. See June 2012 #6b and others.
11) (January 2005 #3b) Suppose $f_n : [0,1] \to \mathbb{R}$ are continuous functions converging uniformly to $f : [0,1] \to \mathbb{R}$. Either prove that $\displaystyle\lim_{n\to\infty} \int_{1/n}^1 f_n(x)\, dx = \int_0^1 f(x)\, dx$ or give a counterexample.

More problems.

12) (January 2006 #7a) Let f be continuous on $[0,1]$ and $f(0) = f(1) = 0$. Show that there is a sequence of polynomials $\{P_n\}$ such that $x(1-x)P_n(x)$ converges to f uniformly.
13) (June 2007 #4b part i) Evaluate $\displaystyle\lim_{n\to\infty} \int_{\pi/2}^{\pi} \frac{n\sin(x/n)}{x}\, dx$ and justify your reasoning.

14) (June 2009 #4a) Let $\{f_n\}$ be a sequence of real-valued continuous functions such that $f_n \to f$ uniformly on $[0,1]$, and let $\{x_n\} \subset [0,1]$ be a sequence which converges to x. Show that $\lim_{n\to\infty} f_n(x_n) = f(x)$.

15) (June 2009 #4b) Prove that the series

$$x^2 + \frac{x^2}{1+x^2} + \frac{x^2}{(1+x^2)^2} + \frac{x^2}{(1+x^2)^3} + \cdots$$

converges uniformly on $[a,\infty)$ for every $a > 0$; but not uniformly on $[0,b]$ for any $b > 0$.

References

[KRD10] Allan P. Donsig Kenneth R. Davidson. *Real analysis and applications.* Springer, 2010.
[Rud76] Walter Rudin. *Principles of mathematical analysis.* McGraw-Hill, Inc., USA, third edition, 1976.