Topology Qual Workshop Day 8: Covering Spaces

Warm-up Problems:

- Explicitly prove that $\mathbb{R}$ is that universal cover of S^1.
- Prove or disprove: $S^2 \vee S^2$ is a covering space of S^2.

(1) (Wisconsin Jan '98) Let $p : \tilde{X} \to X$ be a cover. Suppose that $f, g : Y \to \tilde{X}$ are maps such that $p \circ f$ and $p \circ g$ are equal and assume that f and g agree at $y_0 \in Y$. Show that if Y is connected, then $f = g$.

(2) (Jan '06) Let $p : \tilde{X} \to X$ be a covering map, and $f : Y \to X$ be a continuous map. Define $\tilde{Y} = \left\{(y, \tilde{x}) \in Y \times \tilde{X} : f(y) = p(\tilde{x})\right\} \subseteq Y \times \tilde{X}$, with the subspace topology inherited from $Y \times \tilde{X}$ and define $q : \tilde{Y} \to Y$ by $q(y, \tilde{x}) = y$. Show that q is also a covering map.

(3) (June '05) Show, using covering spaces, that the fundamental group of the Klein bottle is not abelian.

(4) (Purdue Jan '09) Let $p : E \to B$ be a covering map with B connected. Suppose that $p^{-1}(b_0)$ is finite for some $b_0 \in B$. Prove that, for every $b \in B, p^{-1}(b)$ has the same number of elements as $p^{-1}(b_0)$.

(5) (Purdue Aug '09) Let $p : E \to B$ be a covering map. Let Y be locally path-connected. Let $g : Y \to E$ be a function such that

- $p \circ g$ is continuous
- $g \circ \gamma$ is continuous for every path γ in Y.

Prove that g is continuous.

(6) (Purdue Jan '08) Let $p : E \to B$ be a covering map. Suppose that points are closed in B. Let $A \subset E$ be compact. Prove that for every $b \in B$, the set $A \cap p^{-1}(b)$ is finite.

(7) (Purdue Jan '07) Let $p : E \to B$ be a covering map. Prove that p takes open sets to open sets.

(8) Let $G = \mathbb{Z}_2 * \mathbb{Z}_2 = \langle a, b | a^2 = b^2 = 1 \rangle$. Find the presentation complex X_G and the universal cover of X_G, $\tilde{X}_G$. Find an index two subgroup of G and the corresponding covering space.

(9) (June '07) Let $p : \left(\tilde{X}, y_0\right) \to (X, x_0)$ be a covering space projection, $x_0 \in A \subseteq X$, and $\iota : A \to X$ the inclusion map. Show that $q = p|_{p^{-1}(A)} : \left(p^{-1}(A), y_0\right) \to (A, x_0)$ is also a covering space, and $\ker \iota_* \subseteq \operatorname{im}(q_*) \subseteq \pi_1(A, x_0)$.

(10) (June '08) Let $p_i : \left(\tilde{X}_i, \tilde{x}_i\right) \to (X_i, x_i)$ be covering spaces for $i = 1, 2$.

(a) Show that the product space $\tilde{X}_1 \times \tilde{X}_2$ together with the map $p : \tilde{X}_1 \times \tilde{X}_2 \to X_1 \times X_2$ defined by $p(y, z) := (p_1(y), p_2(z))$ is also a covering space.

(b) Find the universal covering space of $S^1 \times D^2 \times S^1$.

(11) (June '09) Let $p : \tilde{X} \to X$ be a covering space. Show that if X is Hausdorff, then $\tilde{X}$ is also Hausdorff.

(12) (May '13) Let X be the space obtained from the 2-sphere S^2 by identifying the north and south poles (i.e. by identifying two diametrically opposite points).

(a) Show that X is homotopy equivalent to $S^1 \vee S^2$.

(b) Describe all connected covering spaces of X.