Topology Ph.D. Qualifying Exam

Gerard Thompson   Mao-Pei Tsui

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This examination has been checked carefully for errors. If you find what you believe to be an error in a question, report this to the proctor. If the proctor's interpretation still seems unsatisfactory to you, you may modify the question so that in your view it is correctly stated, but not in such a way that it becomes trivial. If you feel that the examination is on the long side do not panic. The grading will be adjusted accordingly.

1 Part One: Do six questions

1. If (X, d) is a metric space then $\{x \in X : d(x, x_0) < \epsilon\}$ is said to be the open ball of radius ϵ. Prove that an open ball is an open set.

2. Define what it means for (X, d) to be a metric space. Then $d : X \times X :\to \mathbb{R}$: is d continuous? Discuss. If you cannot answer in general do it for $X = \mathbb{R}$ with the usual topology.

3. A set is said to have the countable-closed topology if the closed sets are the countable sets together with the empty set. If X is a topological space with an uncountably infinite number of points is the diagonal $\Delta := \{(x, x) | x \in X\}$ closed in the finite complement topology? Justify your answer carefully.

4. Let B be an open subset of a topological space X. Prove that a subset $A \subset B$ is relatively open in B if and only if A is open in X.

5. Let D be the open unit disk in the complex plane that is $D := \{z | \, |z| < 1\}$. Let $\sim$ be an equivalence relation on D defined by $z_1 \sim z_2$ if $|z_1| = |z_2|$. Is the quotient space $D/\sim$ Hausdorff? Prove or disprove.

6. Define compactness for a topological space. Prove from your definition that the closed interval $[0, 1]$ is compact.

7. Prove that $\mathbb{R}$ is not compact.

8. Let $X = \Pi_{\mu \in M} X_\mu$ and $Y = \Pi_{\mu \in M} Y_\mu$ be the Cartesian products of the topological spaces $(X_\mu)_{\mu \in M}$ and $(Y_\mu)_{\mu \in M}$ and let X and Y have the product topologies, respectively. Prove that if for each $\mu \in M$ the maps $f_\mu : X_\mu \to Y_\mu$ are continuous then $f : X \to Y$ defined by $f(x)_\mu = f_\mu(x_\mu)$ is continuous.

9. A Hausdorff topological space is known to be *locally compact* if every point has a compact neighborhood. Prove that every closed subspace of a locally compact Hausdorff space is locally compact.

10. Let X be a topological space. Let $A \subset$ X be connected. Prove that the closure $\overline{A}$ of A is connected.

11. Define the term "connected component" for a topological space. Prove that a connected component is connected.

12. A topological space X is said to be *regular* if disjoint singleton and closed sets can be separated by disjoint open sets. Prove that in a regular space disjoint closed and compact sets can be separated by disjoint open sets.

2 Part Two: Do three questions

1. Recall that S^n is defined to be $\{(x_1, x_2, ..., x_n, x_{n+1}) \in \mathbb{R}^{n+1} | x_1^2 + x_2^2 + ... + x_n^2 + x_{n+1}^2 = 1\}$. We shall be interested in the values $n = 1, 2$. Then define $f, g : S^1 \to S^2$ by $f(\cos(s), \sin(s)) = (\cos(s), \sin(s), 0)$ and $g(\cos(s), \sin(s)) = (\cos(s), -\sin(s), 0)$. Show that f is homotopic to g.

2. (i) Let X be a topological space. Let $f, g : I \mapsto X$ be two paths from p to q. Show that $f \sim g$, that is, f is homotopic to g if and only if $f \cdot g^{-1} \sim c_p$ where g^{-1} is the inverse path to g, c_p denotes the constant path based at p and the " $\cdot$ " denotes the product of (compatible) paths.

   (ii) Show that X is simply connected if and only if any two paths in X with the same initial and terminal points are path homotopic.

3. (i) The polygonal symbol of a certain surface without boundary is $xyzx^{-1}zy^{-1}$. Identify the surface. What is its Euler characteristic?

   (ii) Explain how polygons with an even number of sides may be used to classify surfaces without boundary. You do not need to give detailed proofs.

4. Let $X = [0, 1] \times [0, 1]$ denote the rectangle in $\mathbb{R}^2$. Let $\sim$ be the equivalence relation generated by $(0, p) \sim (1, 1 - p)$ wher $0 \leq p \leq 1$. The quotient space $X/\sim$ is called the Möbius band. Show that S^1 is a retract of the Möbius band.

5. Compute the first three homology groups of the *hollow* sphere S^2. You may use simplicial theory and a triangulation but do not simply say that $H_1(S^2)$ is the abelianization of $\pi_1(S^2)$.

6. Compute the fundamental group of the (surface of) a sphere when three points on it are removed.

7. For the sake of this problem a manifold of dimension n will be defined as a topological space in which each point has a neighborhood that is homeomorphic to $\mathbb{R}^n$. If M is a connected manifold of dimension at least 3 and $q \in M$, show that $\pi_1(M - \{q\})$ is isomorphic $\pi_1(M)$.