Department of Mathematics, University of California, Berkeley

STUDENT EXAM NUMBER _____

GRADUATE PRELIMINARY EXAMINATION, Part A Fall Semester 2012

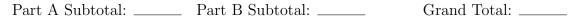
- 1. Please write your 1- or 2-digit student exam number on this cover sheet and on **all** problem sheets (even problems that you do not wish to be graded).
- 2. Indicate below which six problems you wish to have graded. **Cross out** solutions you may have begun for the problems that you have not selected.
- 3. Extra sheets should be stapled to the appropriate problem at the upper right corner. Do not put work for problem p on either side of the page for problem q if $p \neq q$.
- 4. No notes, books, or calculators may be used during the exam.

PROBLEM SELECTION

Part A: List the six problems you have chosen:

GRADE COMPUTATION

1A	1B	Calculus
2A	2B	Real analysis
3A	3B	Real analysis
4A	4B	Complex analysis
5A	5B	Complex analysis
6A	6B	Linear algebra
7A	7B	Linear algebra
8A	8B	Abstract algebra
9A	9B	Abstract algebra



STUDENT EXAM NUMBER _____ Please cross out this problem if you do not wish it graded

Problem 1A. Calculus

Score:

Find the length of the spiral given in polar coordinates by $r = e^{\theta}, -\infty < \theta \leq 0$.

Problem 2A. Real analysis

Score:

Prove or disprove the following assertion:

If $f : \mathbb{R} \to \mathbb{R}$ has the property that f([a, b]) is a bounded closed interval for every $a \leq b$, then f is continuous.

Problem 3A. Real analysis

Score:

Prove the existence of the limit

$$\lim_{n \to \infty} \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{n} - \log n.$$

Problem 4A. Complex analysis

Score:

(a) Find the poles and residues of $1/(z^3 \cos(z))$.

(b) Show that the integral of the function above over a square contour centered at the origin with side $2\pi N$ tends to zero as the integer N tends to infinity.

(c) Find the sum $1/1^3 - 1/3^3 + 1/5^3 - 1/7^3 + \cdots$.

Problem 5A. Complex analysis

Score:

(a) Show that if |z| < 1 then there is a holomorphic function defined on some neighborhood of the unit disk whose only zero is at z and that has absolute value 1 on the unit circle.

(b) Suppose that f is a holomorphic function on the complex plane and is not identically zero. Show that there is a holomorphic function g defined in some open set containing the unit disk such that |f(z)| = |g(z)| whenever |z| = 1, and such that g has no zeros in the open unit disk.

Problem 6A. Linear algebra

Score:

If U, V, W are subspaces of a vector space such that any two have intersection zero, prove that

 $\dim(U+V+W) + \dim U + \dim V + \dim W \le \dim(U+V) + \dim(V+W) + \dim(W+U)$

and give an example where equality does not hold.

Problem 7A. Linear algebra

Score:

Let I_n denote the $n \times n$ identity matrix, and J_n the $n \times n$ matrix with all entries equal to 1. Determine for which real numbers a the matrix $I_n + aJ_n$ is invertible, and find its inverse.

Score:

Let G be a finite Abelian group of order n. Suppose m is a square-free (not divisible by the square of a prime), positive integer dividing n. Show that G contains an element of order m. Give an example to show that this need not be true if m is not assumed to be square-free.

Problem 9A. Abstract algebra

Score:

Does there exists a homomorphism of commutative rings with unit from $\mathbb{Z}[x]/(x^2+3)$ to $\mathbb{Z}[x]/(x^2-x+1)$? Either exhibit such a homomorphism, or prove that none exists.

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Fall Semester 2012

GRADUATE PRELIMINARY EXAMINATION, Part B

- 1. Please write your 1- or 2-digit student exam number on this cover sheet and on **all** problem sheets (even problems that you do not wish to be graded).
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4. No notes, books, or calculators may be used during the exam.

PROBLEM SELECTION

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Part B: List the six problems you have chosen:

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Problem 1B. Calculus

Prove that

$$\frac{\pi}{4} = 4\arctan\frac{1}{5} - \arctan\frac{1}{239}.$$

In 1706 John Machin used this formula to calculate π to 100 decimal places. Explain briefly why he did not use the simpler formula $\frac{\pi}{4} = \arctan 1$.

Solution:

Score:

Problem 2B. Real analysis

Score:

- (a) Find the sum $1 1/2 + 1/3 1/4 + \cdots$
- (b) Find the sum $1 1/2 1/4 + 1/3 1/6 1/8 + 1/5 1/10 1/12 + \cdots$

Problem 3B. Real analysis

Score:

Let (X, d_X) and (Y, d_Y) be metric spaces. Suppose that a map $\pi : X \to Y$ is a *submetry*; this means that for every $x \in X$ and any r > 0, the image of the closed r-ball around x is the closed r-ball around $\pi(x)$.

(a) Show that π is surjective if X is nonempty.

(b) Show that π is continuous.

(c) Show that π is open (meaning that the image of any open subset is open). Solution:

Problem 4B. Complex analysis

Compute

 $\int_0^\infty \frac{dx}{(x^2+1)^2}.$

Solution:

Score:

Problem 5B. Complex analysis

Score:

Show that as the positive integer N tends to infinity, the change in argument of $e^z - z$ is bounded on 3 sides of the square with corners $\pm 2\pi N \pm 2\pi i N$ but is unbounded on the fourth side. Show that $e^z = z$ has infinitely many complex roots.

Problem 6B. Linear algebra

Score:

Find all eigenvalues and eigenvectors of the linear map $T: \mathbb{C}^n \to \mathbb{C}^n$ given by $T((x_1, \ldots, x_n)) = (x_2, x_3, \ldots, x_n, x_1).$

Problem 7B. Linear algebra

Score:

Suppose that A and B are linear transformations of a finite dimensional complex vector space such that AB - BA = A. If v is an eigenvector of B with eigenvalue λ , show that Av is zero or an eigenvector of B and find its eigenvalue. Prove that A is nilpotent.

Problem 8B. Abstract algebra

Score:

Let R be a commutative ring with unit. Suppose that there is a monic polynomial $p(x) \in R[x]$ such that the ideal $(p(x)) \subseteq R[x]$ is maximal. Prove that R is a field.

Problem 9B.	Abstract	algebra
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Score:

Let M be a (possibly singular) square matrix over a field F. Let p be the product of the nonzero eigenvalues of M (counted with multiplicities) in some algebraically closed extension Kof F. Prove that $p \in F$.