Department of Mathematics, University of California, Berkeley

STUDENT EXAM NUMBER ________________

GRADUATE PRELIMINARY EXAMINATION, Part A — Fall Semester 2013

---

1. Please write your 1- or 2-digit student exam number on this cover sheet and on **all** problem sheets (even problems that you do not wish to be graded).
2. Indicate below which six problems you wish to have graded. **Cross out** solutions you may have begun for the problems that you have not selected.
3. Extra sheets should be stapled to the appropriate problem at the upper right corner. Do not put work for problem p on either side of the page for problem q if $p \neq q$.
4. No notes, books, or calculators may be used during the exam.

PROBLEM SELECTION

Part A: List the six problems you have chosen:

______ , ______ , ______ , ______ , ______ , ______

GRADE COMPUTATION

1A. ______	1B. ______	Calculus
2A. ______	2B. ______	Real analysis
3A. ______	3B. ______	Real analysis
4A. ______	4B. ______	Complex analysis
5A. ______	5B. ______	Complex analysis
6A. ______	6B. ______	Linear algebra
7A. ______	7B. ______	Linear algebra
8A. ______	8B. ______	Abstract algebra
9A. ______	9B. ______	Abstract algebra

Part A Subtotal: ______ Part B Subtotal: ______ Grand Total: ______

STUDENT EXAM NUMBER ________________________

Please cross out this problem if you do not wish it graded

Problem 1A. *Score:*

The set of pairs of positive real numbers (x, y) with $x^y = y^x$ is a union of two smooth curves. Find the point where they intersect.

Solution:

STUDENT EXAM NUMBER ________________________
Please cross out this problem if you do not wish it graded

Problem 2A. *Score:*

Suppose that x is a smooth real-valued function of the real number t, satisfying $dx/dt \leq b(t)x(t)$ for some continuous function b. Prove that if $s \leq t$ then $x(t) \leq x(s) \exp \int_s^t b(t)dt$.

Solution:

STUDENT EXAM NUMBER __________________________

Please cross out this problem if you do not wish it graded

Problem 3A. *Score:*

Define a set of positive real numbers as follows. Let $x_0 > 0$ be any positive number, and let $x_{n+1} = (1 + x_n)^{-1}$ for all $n \geq 0$. Prove that this sequence converges, and find its limit.

Solution:

STUDENT EXAM NUMBER ______________________

Please cross out this problem if you do not wish it graded

Problem 4A. *Score:*

Show that the polynomial $p(z) = z^5 - 6z + 3$ has five distinct complex roots, and that exactly three (and not five) are real.

Solution:

STUDENT EXAM NUMBER ________________________

Please cross out this problem if you do not wish it graded

Problem 5A. *Score:*

Compute
$$\int_0^{2\pi} \frac{\cos(x)}{2+\cos(x)}\, dx.$$

Solution:

STUDENT EXAM NUMBER ______________________
Please cross out this problem if you do not wish it graded

Problem 6A. *Score:*

Let V be the complex vector space of complex 2×2 matrices X. Find all quadratic forms Q on V such that $Q(X) = Q(AXA^{-1})$ for any complex invertible 2×2 matrix A.

Solution:

STUDENT EXAM NUMBER ________________________

Please cross out this problem if you do not wish it graded

Problem 7A. *Score:*

Let A and B be $n \times n$ complex matrices. Prove or disprove each of the following statements:

1. If A and B are diagonalizable, so is $A + B$.
2. If A and B are diagonalizable, so is AB.
3. If $A^2 = A$, then A is diagonalizable.
4. If A^2 is diagonalizable, then A is diagonalizable.

Solution:

STUDENT EXAM NUMBER ______________________

Please cross out this problem if you do not wish it graded

Problem 8A. *Score:*

Let R be a (possibly non-commutative) ring with identity, and let u be an element of R with a right inverse. Prove that the following conditions on u are equivalent:

1. u has more than one right inverse;
2. u is a zero divisor;
3. u is not a unit.

Solution:

STUDENT EXAM NUMBER ________________________

Please cross out this problem if you do not wish it graded

Problem 9A. *Score:*

Give an example of a group G such that the center of G modulo its center is non-trivial. Give an example of a group H such that the groups H, H', H'' and H''' are all distinct. (The derived group H' of a group H is the subgroup generated by commutators, or equivalently the smallest subgroup such that the quotient by this subgroup is an abelian group.)

Solution:

Department of Mathematics, University of California, Berkeley

STUDENT EXAM NUMBER ________________

GRADUATE PRELIMINARY EXAMINATION, Part B Fall Semester 2013

---

1. Please write your 1- or 2-digit student exam number on this cover sheet and on **all** problem sheets (even problems that you do not wish to be graded).
2. Indicate below which six problems you wish to have graded. **Cross out** solutions you may have begun for the problems that you have not selected.
3. Extra sheets should be stapled to the appropriate problem at the upper right corner. Do not put work for problem p on either side of the page for problem q if $p \neq q$.
4. No notes, books, or calculators may be used during the exam.

PROBLEM SELECTION

Part B: List the six problems you have chosen:

______ , ______ , ______ , ______ , ______ , ______

STUDENT EXAM NUMBER ________________________

Please cross out this problem if you do not wish it graded

Problem 1B. *Score:*

For which pairs of real numbers (a, b) does the series $\sum_{n=3}^{\infty} n^a (\log n)^b$ converge?

Solution:

STUDENT EXAM NUMBER ______________________
Please cross out this problem if you do not wish it graded

Problem 2B. *Score:*

Say that a metric space X has property (A) if the image of every continuous function $f : X \to \mathbf{R}$ is an interval, which may be open, closed or half-open. Prove that X has property (A) if and only if it is connected.

Solution:

STUDENT EXAM NUMBER ________________________

Please cross out this problem if you do not wish it graded

Problem 3B. *Score:*

Suppose that $f : (0,1) \to \mathbf{R}$ is a continuous function with $\int_0^1 |f(t)|\, dt < \infty$. Define $g : (0,1) \to \mathbf{R}$ by

$$g(x) = \int_x^1 \frac{f(t)}{t}\, dt.$$

Show that $\int_0^1 |g(x)|\, dx < \infty$.

Solution:

STUDENT EXAM NUMBER ________________________

Please cross out this problem if you do not wish it graded

Problem 4B. *Score:*

Compute

$$\lim_{N\to\infty}\int_{-N}^{N}\frac{x\sin(x)}{x^2+1}\,dx.$$

Solution:

STUDENT EXAM NUMBER ____________________

Please cross out this problem if you do not wish it graded

Problem 5B. *Score:*

Let $U \subset \mathbf{C}$ be a bounded open set containing 0, and let $f : U \to U$ be an analytic function, whose Taylor series at 0 is

$$f(z) = z + a_2 z^2 + a_3 z^3 + \dots$$

Prove that $a_2 = 0$. (Hint : consider the functions $g_n(z) = f \circ \ldots \circ f(z)$ obtained by composing f with itself n times.)

Solution:

STUDENT EXAM NUMBER ____________________

Please cross out this problem if you do not wish it graded

Problem 6B. *Score:*

Is it possible to find two real 2×2 matrices A, B such that $A^2 = B^2 = Id$ (the identity matrix), but AB has eigenvalues 2 and 1/2?

Solution:

STUDENT EXAM NUMBER ______________________

Please cross out this problem if you do not wish it graded

Problem 7B. *Score:*

Suppose that A is an m by n complex matrix and B is an n by m complex matrix, and write I_m for the m by m identity matrix. Show that if $I_m - AB$ is invertible then so is $I_n - BA$. (Hint: what does the condition that $I_m - X$ is not invertible say about eigenvalues and eigenvectors of X?)

Solution:

STUDENT EXAM NUMBER ________________________

Please cross out this problem if you do not wish it graded

Problem 8B. *Score:*

Prove that if n is coprime to $N = 561$ then $n^{N-1} \equiv 1 \bmod N$.

Solution:

STUDENT EXAM NUMBER ______________________

Please cross out this problem if you do not wish it graded

Problem 9B. *Score:*

How many irreducible polynomials of degree exactly 6 are there over the finite field with 3 elements?

Solution: