Department of Mathematics, University of California, Berkeley

STUDENT EXAM NUMBER

GRADUATE PRELIMINARY EXAMINATION, Part A Fall Semester 2013

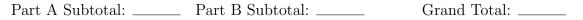
- 1. Please write your 1- or 2-digit student exam number on this cover sheet and on **all** problem sheets (even problems that you do not wish to be graded).
- 2. Indicate below which six problems you wish to have graded. **Cross out** solutions you may have begun for the problems that you have not selected.
- 3. Extra sheets should be stapled to the appropriate problem at the upper right corner. Do not put work for problem p on either side of the page for problem q if $p \neq q$.
- 4. No notes, books, or calculators may be used during the exam.

PROBLEM SELECTION

Part A: List the six problems you have chosen:

GRADE COMPUTATION

1A	1B	Calculus
2A	2B	Real analysis
3A	3B	Real analysis
4A	4B	Complex analysis
5A	5B	Complex analysis
6A	6B	Linear algebra
7A	7B	Linear algebra
8A	8B	Abstract algebra
9A	9B	Abstract algebra



Problem 1A.

Score:

The set of pairs of positive real numbers (x, y) with $x^y = y^x$ is a union of two smooth curves. Find the point where they intersect.

Problem 2A.

Score:

Suppose that x is a smooth real-valued function of the real number t, satisfying $dx/dt \leq b(t)x(t)$ for some continuous function b. Prove that if $s \leq t$ then $x(t) \leq x(s) \exp \int_s^t b(t) dt$.

Problem 3A.

Score:

Define a set of positive real numbers as follows. Let $x_0 > 0$ be any positive number, and let $x_{n+1} = (1 + x_n)^{-1}$ for all $n \ge 0$. Prove that this sequence converges, and find its limit.

Problem 4A.

Score:

Show that the polynomial $p(z) = z^5 - 6z + 3$ has five distinct complex roots, and that exactly three (and not five) are real.

Problem 5A.

Compute

$$\int_0^{2\pi} \frac{\cos(x)}{2 + \cos(x)} \, dx.$$

Solution:

Score:

Problem 6A.

Score:

Let V be the complex vector space of complex 2×2 matrices X. Find all quadratic forms Q on V such that $Q(X) = Q(AXA^{-1})$ for any complex invertible 2×2 matrix A.

Problem 7A.

Score:

Let A and B be $n \times n$ complex matrices. Prove or disprove each of the following statements:

- 1. If A and B are diagonalizable, so is A + B.
- 2. If A and B are diagonalizable, so is AB.
- 3. If $A^2 = A$, then A is diagonalizable.
- 4. If A^2 is diagonalizable, then A is diagonalizable.

Problem 8A.

Score:

Let R be a (possibly non-commutative) ring with identity, and let u be an element of R with a right inverse. Prove that the following conditions on u are equivalent:

- 1. u has more than one right inverse;
- 2. u is a zero divisor;
- 3. u is not a unit.

Problem 9A.

Score:

Give an example of a group G such that the center of G modulo its center is non-trivial. Give an example of a group H such that the groups H, H', H'' and H''' are all distinct. (The derived group H' of a group H is the subgroup generated by commutators, or equivalently the smallest subgroup such that the quotient by this subgroup is an abelian group.)

STUDENT EXAM NUMBER _____

GRADUATE PRELIMINARY EXAMINATION, Part B

Fall Semester 2013

- 1. Please write your 1- or 2-digit student exam number on this cover sheet and on **all** problem sheets (even problems that you do not wish to be graded).
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4. No notes, books, or calculators may be used during the exam.

PROBLEM SELECTION

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Part B: List the six problems you have chosen:

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_ , _

_ , __

Problem 1B.

Score:

For which pairs of real numbers (a, b) does the series $\sum_{n=3}^{\infty} n^a (\log n)^b$ converge?

Problem 2B.

Score:

Say that a metric space X has property (A) if the image of every continuous function $f: X \to \mathbf{R}$ is an interval, which may be open, closed or half-open. Prove that X has property (A) if and only if it is connected.

Problem 3B.

Score:

Suppose that $f: (0,1) \to \mathbf{R}$ is a continuous function with $\int_0^1 |f(t)| dt < \infty$. Define $g: (0,1) \to \mathbf{R}$ by

$$g(x) = \int_x^1 \frac{f(t)}{t} \, dt.$$

Show that $\int_0^1 |g(x)| dx < \infty$.

Problem 4B.

Compute

$$\lim_{N \to \infty} \int_{-N}^{N} \frac{x \sin(x)}{x^2 + 1} \, dx.$$

Solution:

Score:

Problem 5B.

Score:

Let $U \subset \mathbf{C}$ be a bounded open set containing 0, and let $f : U \to U$ be an analytic function, whose Taylor series at 0 is

$$f(z) = z + a_2 z^2 + a_3 z^3 + \dots$$

Prove that $a_2 = 0$. (Hint : consider the functions $g_n(z) = f \circ \ldots \circ f(z)$ obtained by composing f with itself n times.)

Problem 6B.

Score:

Is it possible to find two real 2×2 matrices A, B such that $A^2 = B^2 = Id$ (the identity matrix), but AB has eigenvalues 2 and 1/2?

Problem 7B.

Score:

Suppose that A is an m by n complex matrix and B is an n by m complex matrix, and write I_m for the m by m identity matrix. Show that if $I_m - AB$ is invertible then so is $I_n - BA$. (Hint: what does the condition that $I_m - X$ is not invertible say about eigenvalues and eigenvectors of X?)

Problem 8B.

Score:

Prove that if n is coprime to N = 561 then $n^{N-1} \equiv 1 \mod N$.

Problem 9B.

Score:

How many irreducible polynomials of degree exactly 6 are there over the finite field with 3 elements?