Department of Mathematics, University of California, Berkeley

STUDENT EXAM NUMBER ________

GRADUATE PRELIMINARY EXAMINATION, Part A — Fall Semester 2013

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1. Please write your 1- or 2-digit student exam number on this cover sheet and on **all** problem sheets (even problems that you do not wish to be graded).
2. Indicate below which six problems you wish to have graded. **Cross out** solutions you may have begun for the problems that you have not selected.
3. Extra sheets should be stapled to the appropriate problem at the upper right corner. Do not put work for problem p on either side of the page for problem q if $p \neq q$.
4. No notes, books, or calculators may be used during the exam.

PROBLEM SELECTION

Part A: List the six problems you have chosen:

____ , ____ , ____ , ____ , ____ , ____

GRADE COMPUTATION

1A. ____	1B. ____	Calculus
2A. ____	2B. ____	Real analysis
3A. ____	3B. ____	Real analysis
4A. ____	4B. ____	Complex analysis
5A. ____	5B. ____	Complex analysis
6A. ____	6B. ____	Linear algebra
7A. ____	7B. ____	Linear algebra
8A. ____	8B. ____	Abstract algebra
9A. ____	9B. ____	Abstract algebra

Part A Subtotal: ____ Part B Subtotal: ____ Grand Total: ____

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Please cross out this problem if you do not wish it graded

Problem 1A. *Score:*

The set of pairs of positive real numbers (x, y) with $x^y = y^x$ is a union of two smooth curves. Find the point where they intersect.

Solution:

The set is the set of points with $x^{1/x} = y^{1/y}$. The graph of the function $f(x) = x^{1/x}$ rises from $(0, 0)$ to a maximum at $(e, e^{1/e})$ and then decreases to $(\infty, 1)$. So the points with $x^y = y^x$ consists of the line $y = x$ together with the pairs of distinct points having the same value of $f(x)$, and these curves meet whenever f has a maximum or minumum value, so the intersection point is (e, e).

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Problem 2A. *Score:*

Suppose that x is a smooth real-valued function of the real number t, satisfying $dx/dt \leq b(t)x(t)$ for some continuous function b. Prove that if $s \leq t$ then $x(t) \leq x(s) \exp \int_s^t b(t)dt$.

Solution: Put $y(t) = x(t) \exp(-\int_s^t b(t)dt)$. Then the condition $dx/dt \leq b(t)x(t)$ implies that $dy/dt \leq 0$, so $y(t) \leq y(s)$, which is equivalent to the result.

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Problem 3A. *Score:*

Define a set of positive real numbers as follows. Let $x_0 > 0$ be any positive number, and let $x_{n+1} = (1 + x_n)^{-1}$ for all $n \geq 0$. Prove that this sequence converges, and find its limit.

Solution:

First of all, since $x_0 > 0$, $1 + x_0 > 1$ and therefore $0 < x_1 < 1$. Similarly, $1 < 1 + x_1 < 2$, so $\frac{1}{2} < x_2 < 1$. Finally, $\frac{3}{2} < 1 + x_2 < 2$, so $\frac{1}{2} < x_3 < \frac{2}{3}$. By induction, we then have $\frac{1}{2} < x_n < \frac{2}{3}$ for all $n \geq 3$.

Let $f(x) = 1/(1 + x)$. Solving for $f(x) = x$ leads to an equation $x^2 + x - 1 = 0$, so $\alpha := (\sqrt{5} - 1)/2$ is the unique positive real number for which $f(\alpha) = \alpha$.

We have $f'(x) = -1/(1 + x)^2$, so

$$|f'(x)| < \frac{4}{9} \qquad \text{for all } \frac{1}{2} < x < \frac{2}{3}\ .$$

Since α lies in the given range (apply the first paragraph with $x_0 = \alpha$), we have by the Mean Value Theorem that

$$|x_{n+1} - \alpha| = |f(x_n) - f(\alpha)| < \frac{4}{9}|x_n - \alpha|$$

for all $n \geq 3$, so the sequence $\{x_n\}$ converges to $\alpha = \frac{\sqrt{5}-1}{2}$ as $n \to \infty$.

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Problem 4A. *Score:*

Show that the polynomial $p(z) = z^5 - 6z + 3$ has five distinct complex roots, and that exactly three (and not five) are real.

Solution:

We know that p has five roots, when counted with multiplicity, so it suffices for the first part to show that p has no multiple roots. Any multiple root of p must be a common root of p and p', hence it must be a root of

$$5p(z) - zp'(z) = (5z^5 - 30z + 15) - (5z^5 - 6z) = -24z + 15 \ .$$

Thus, the only possible multiple root of p is $z = 5/8$. But this is clearly seen not to be a root of p', since $5(5/8)^4 - 6 \neq 0$. Thus p has five distinct complex roots.

To count the real roots, we note that $p(-2) = -17 < 0$, $p(0) = 3 > 0$, $p(1) = -2 < 0$, and $p(2) = 23 > 0$. So, by continuity, p has at least three real roots. However, $p'(x) = 5x^4 - 6$ is negative when $|x| < \sqrt[4]{6/5}$ and positive when $|x| > \sqrt[4]{6/5}$, so it has at most one root on each of the three intervals

$$(-\infty, -\sqrt[4]{6/5}), \quad (-\sqrt[4]{6/5}, \sqrt[4]{6/5}), \quad (\sqrt[4]{6/5}, \infty) \ .$$

Therefore p has exactly three real roots.

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Problem 5A. *Score:*

Compute

$$\int_0^{2\pi} \frac{\cos(x)}{2+\cos(x)}\,dx.$$

Solution:

Put $z = e^{ix}$. Then

$$\int_0^{2\pi} \frac{\cos(x)}{2+\cos(x)}\,dx = \int_{|z|=1} \frac{\frac{z+z^{-1}}{2}}{2+\frac{z+z^{-1}}{2}}\,\frac{dz}{iz} = \int_{|z|=1} \frac{z^2+1}{z(z^2+4z+1)}\,\frac{dz}{i}.$$

The poles inside the unit circle are at $z=0$ and $z=\sqrt{3}-2$. The answer is

$$2\pi\left(Res_{z=0}\frac{z^2+1}{z(z^2+4z+1)} + Res_{z=\sqrt{3}-2}\frac{z^2+1}{z(z^2+4z+1)}\right) =$$

$$2\pi\left(1+\frac{4-2\sqrt{3}}{3-2\sqrt{3}}\right) = 2\pi\cdot\frac{7-4\sqrt{3}}{3-2\sqrt{3}}.$$

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Problem 6A. *Score:*

Let V be the complex vector space of complex 2×2 matrices X. Find all quadratic forms Q on V such that $Q(X) = Q(AXA^{-1})$ for any complex invertible 2×2 matrix A.

Solution:

Take $Q(X) = xTr(X^2) + yTr(X)^2$. It clearly satisfies the requirements. These are the only possibilities, as the condition of invariance implies that Q is determined by its values on diagonal matrices (as diagonalizable matrices are dense in all matrices). Also Q is invariant under exchanging the two diagonal entries of a diagonal matrix, from which it follows easily that the space of invariant forms is at most 2-dimensional. (A similar argument works for n by n matrices, except that if $n < 2$ the space of quadratic forms has dimension less than 2 due to degeneracies.)

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Problem 7A. *Score:*

Let A and B be $n \times n$ complex matrices. Prove or disprove each of the following statements:

1. If A and B are diagonalizable, so is $A + B$.
2. If A and B are diagonalizable, so is AB.
3. If $A^2 = A$, then A is diagonalizable.
4. If A^2 is diagonalizable, then A is diagonalizable.

Solution:

1, 2, 4 are false, with many 2 by 2 counterexamples using the fact that any matrix of the form $\binom{ab}{0a}$ with b non-zero is not diagonalizable. 3 is true, as the minimal polynomial of A has no repeated roots.

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Problem 8A. *Score:*

Let R be a (possibly non-commutative) ring with identity, and let u be an element of R with a right inverse. Prove that the following conditions on u are equivalent:

1. u has more than one right inverse;
2. u is a zero divisor;
3. u is not a unit.

Solution:

Let v be a right inverse of u: $uv = 1$.

(1) $\implies$ (2): If v' is another right inverse of u, then $u(v - v') = 1 - 1 = 0$ with $v - v' \neq 0$, so u is a zero divisor.

(2) $\implies$ (3): We prove the contrapositive. If u is a unit and if $uw = 0$, then $0 = u^{-1} \cdot 0 = u^{-1}uw = w$, so $w = 0$. Similarly, $wu = 0$ implies $w = 0$. Thus u is not a zero divisor.

(3) $\implies$ (1): Since u is not a unit, $vu \neq 1$. Let $v' = v + (vu - 1)$. Then $v' \neq v$, and

$$uv' = u(v + vu - 1) = uv + uvu - u = 1 + 1u - u = 1\ .$$

Thus v' is another right inverse of u.

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Problem 9A. *Score:*

Give an example of a group G such that the center of G modulo its center is non-trivial. Give an example of a group H such that the groups H, H', H'' and H''' are all distinct. (The derived group H' of a group H is the subgroup generated by commutators, or equivalently the smallest subgroup such that the quotient by this subgroup is an abelian group.)

Solution:

The group G can be the dihedral group of order 8. The group H can be the symmetric group S_4 on 4 points, with derived groups the alternating group A_4, the Klein 4-group, and the trivial group.

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GRADUATE PRELIMINARY EXAMINATION, Part B — Fall Semester 2013

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PROBLEM SELECTION

Part B: List the six problems you have chosen:

______ , ______ , ______ , ______ , ______ , ______

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Problem 1B. *Score:*

For which pairs of real numbers (a, b) does the series $\sum_{n=3}^{\infty} n^a (\log n)^b$ converge?

Solution:

By the integral test this is equivalent to asking for convergence of the integra l

$$\int_{x=3}^{\infty} x^a (\log x)^b dx$$

This converges if $a < -1$ and diverges if $a > -1$ by comparison with $\int x^s dx$. If $a = -1$ then it converges for $b < -1$ and diverges if $b > -1$ again by doing t he integral explicitly, using the fact that the derivative of $(\log x)^{b+1}$ i s $(b + 1)(\log x)^b x^{-1}$. For $a = b = -1$ it diverges as the derivative of $\log \log x$ is $x^{-1}(\log x)^{-1}$.

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Problem 2B. *Score:*

Say that a metric space X has property (A) if the image of every continuous function $f : X \to \mathbf{R}$ is an interval, which may be open, closed or half-open. Prove that X has property (A) if and only if it is connected.

Solution:

Suppose that X does not have property (A). Then there is a continuous function $f : X \to \mathbf{R}$ whose image is not an interval. If X is connected then $f(X)$ is a connected subset of $\mathbf{R}$, which must be an interval. Hence X is not connected.

Suppose that X is not connected. Write $X = X_1 \cup X_2$, where X_1 and X_2 are nonempty disjoint open subsets. Define $f : X \to \mathbf{R}$ to be one on X_1 and two on X_2. Then f is continuous and its image is not an interval. Hence X does not have property (A).

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Problem 3B. *Score:*

Suppose that $f : (0,1) \to \mathbf{R}$ is a continuous function with $\int_0^1 |f(t)|\,dt < \infty$. Define $g : (0,1) \to \mathbf{R}$ by

$$g(x) = \int_x^1 \frac{f(t)}{t}\,dt.$$

Show that $\int_0^1 |g(x)|\,dx < \infty$.

Solution:

For any $\epsilon > 0$,

$$\begin{aligned}\int_\epsilon^1 |g(x)|\,dx &= \int_\epsilon^1 \left|\int_x^1 \frac{f(t)}{t}\,dt\right|\,dx \le \int_\epsilon^1 \int_x^1 \frac{|f(t)|}{t}\,dt\,dx \\ &= \int_\epsilon^1 \int_\epsilon^t \frac{|f(t)|}{t}\,dx\,dt = \int_\epsilon^1 \left(1 - \frac{\epsilon}{t}\right) |f(t)|\,dt \le \int_0^1 |f(t)|\,dt.\end{aligned}$$

Hence $\int_0^1 |g(x)|\,dx < \infty$.

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Problem 4B. *Score:*

Compute

$$\lim_{N\to\infty}\int_{-N}^{N}\frac{x\sin(x)}{x^2+1}\,dx.$$

Solution:

Let C_N be the circular arc in the upper half plane of radius N around the origin, oriented counterclockwise. For N large.

$$\int_{-N}^{N}\frac{xe^{ix}}{x^2+1}\,dx+\int_{C_N}\frac{ze^{iz}}{z^2+1}\,dz=2\pi i\;Res_{z=i}\frac{ze^{iz}}{z^2+1}=\frac{\pi}{e}i.$$

By Jordan's Lemma, $\lim_{N\to\infty}\int_{C_N}\frac{ze^{iz}}{z^2+1}\,dz=0$. Taking imaginary parts gives

$$\lim_{N\to\infty}\int_{-N}^{N}\frac{x\sin(x)}{x^2+1}\,dx=\frac{\pi}{e}.$$

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Problem 5B. *Score:*

Let $U \subset \mathbf{C}$ be a bounded open set containing 0, and let $f : U \to U$ be an analytic function, whose Taylor series at 0 is

$$f(z) = z + a_2 z^2 + a_3 z^3 + \ldots$$

Prove that $a_2 = 0$. (Hint : consider the functions $g_n(z) = f \circ \ldots \circ f(z)$ obtained by composing f with itself n times.)

Solution:

Suppose that U is contained in $\{z : |z| \leq R\}$ for some $R < \infty$. Then $|g_n(z)| \leq R$ for all $z \in U$. The Taylor series expansion of g_n is $g_n(z) = z + na_2 z^2 + \ldots$. If U contains $\overline{B(0, \epsilon)}$ then

$$na_2 = \frac{1}{2\pi i} \int_{|z|=\epsilon} \frac{g_n(z)}{z^3}\, dz$$

and $n|a_2| \leq \frac{R}{\epsilon^2}$. Taking $n \to \infty$ implies that $a_2 = 0$.

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Problem 6B. *Score:*

Is it possible to find two real 2×2 matrices A, B such that $A^2 = B^2 = Id$ (the identity matrix), but AB has eigenvalues 2 and 1/2?

Solution:

Yes. Take

$$A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad B = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

Then require $trB = a + d = 0$, $detB = ad - bc = -1$, so $d = -a$. Then require $trAB = 2a = 5/2$, $detAB = a^2 + bc = 1$. Solve for example by

$$B = \begin{pmatrix} 5/4 & 1 \\ -21/4 & -5/4 \end{pmatrix}$$

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Problem 7B. *Score:*

Suppose that A is an m by n complex matrix and B is an n by m complex matrix, and write I_m for the m by m identity matrix. Show that if $I_m - AB$ is invertible then so is $I_n - BA$. (Hint: what does the condition that $I_m - X$ is not invertible say about eigenvalues and eigenvectors of X?)

Solution:

$I_m - AB$ fails to be invertible exactly when it has an eigenvalue 0, in other words when there is a nonzero vector v in R^m fixed by AB. But then Bv is a vector in R^n fixed by BA, so $I_n - BA$ fails to be invertible.

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Problem 8B. *Score:*

Prove that if n is coprime to $N = 561$ then $n^{N-1} \equiv 1 \bmod N$.

Solution:

The prime factors p of N are 3, 11, and 17 (each of which divides it to just the first power). For each prime factor p of N, $N - 1$ is divisible by $p - 1$ so $n^{N-1} \equiv 1 \bmod p$ by Fermat's theorem. So $n^{N-1} \equiv 1 \bmod N$.

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Problem 9B. *Score:*

How many irreducible polynomials of degree exactly 6 are there over the finite field with 3 elements?

Solution: $2(3^6 - 3^3 - 3^2 + 3^1)/6$. The first factor of 2 is the possible leading coefficients, the second factor is the number of elements of the field of order 3^6 not in any smaller field, and the third factor of $1/6$ comes because each irreducible polynomial of degree 6 has 6 distinct roots generating this field.