Department of Mathematics, University of California, Berkeley

YOUR 1 OR 2 DIGIT EXAM NUMBER ____

GRADUATE PRELIMINARY EXAMINATION, Part A — Fall Semester 2015

---

1. Please write your 1- or 2-digit exam number on this cover sheet and on **all** problem sheets (even problems that you do not wish to be graded).
2. Indicate below which six problems you wish to have graded. **Cross out** solutions you may have begun for the problems that you have not selected.
3. Extra sheets should be stapled to the appropriate problem at the upper right corner. Do not put work for problem p on either side of the page for problem q if $p \neq q$.
4. No notes, books, calculators, or other electronic devices may be used during the exam.

PROBLEM SELECTION

Part A: List the six problems you have chosen:

______ , ______ , ______ , ______ , ______ , ______

GRADE COMPUTATION

1A. ______	1B. ______	Calculus
2A. ______	2B. ______	Real analysis
3A. ______	3B. ______	Real analysis
4A. ______	4B. ______	Complex analysis
5A. ______	5B. ______	Complex analysis
6A. ______	6B. ______	Linear algebra
7A. ______	7B. ______	Linear algebra
8A. ______	8B. ______	Abstract algebra
9A. ______	9B. ______	Abstract algebra

Part A Subtotal: ______ Part B Subtotal: ______ Grand Total: ______

YOUR EXAM NUMBER ____

Please cross out this problem if you do not wish it graded

Problem 1A. *Score:*

Find the real values of x for which

$$\sum_{n=0}^{\infty} \frac{(1/2)(3/2)\cdots((2n-1)/2)}{n!}\Big(\frac{2x}{1+x^2}\Big)^{2n} = 1 + \frac{1}{2}\Big(\frac{2x}{1+x^2}\Big)^2 + \frac{1}{2}\frac{3}{4}\Big(\frac{2x}{1+x^2}\Big)^4 + \cdots$$

converges and sum it for these numbers. Caution: there is something unusual about the sum of this series.

Solution:

YOUR EXAM NUMBER ____

Please cross out this problem if you do not wish it graded

Problem 2A. *Score:*

Show that there is a real-valued function on the real plane that is not continuous, but is continuous when restricted to any straight line.

Solution:

YOUR EXAM NUMBER ____

Please cross out this problem if you do not wish it graded

Problem 3A. *Score:*

Let $f(x)$ be differentiable on an interval (a, b).

(a) Prove that if X is the range of $(f(u) - f(v))/(u - v)$ for $a < u < v < b$ and Y is the range of $f'(x)$ on (a, b) then $X \subseteq Y \subseteq \overline{X}$.

(b) Prove that the range of $f'(x)$ on (a, b) is an interval (possibly unbounded). Do not assume that $f'(x)$ is continuous.

Solution:

YOUR EXAM NUMBER ____

Please cross out this problem if you do not wish it graded

Problem 4A. *Score:*

Let $S = \mathbb{C} \cup \{\infty\}$ be the Riemann sphere. Let $\phi\colon \mathbb{C}^2 \setminus \{(0,0)\} \to S$ be the map defined by $\phi(w,z) = w/z$ for $z \neq 0$, $\phi(w,0) = \infty$.

(a) Prove that there is a unique map $\tau\colon S \to S$ with the following property: $\tau(\phi(w,z)) = \phi(w',z')$ if and only if the one-dimensional subspaces $\mathbb{C}\cdot(w,z)$ and $\mathbb{C}\cdot(w',z')$ are orthogonal under the standard Hermitian inner product on $\mathbb{C}^2$ in which the unit vectors $(1,0)$ and $(0,1)$ are orthonormal.

(b) Prove that τ is continuous and bijective.

(c) Determine, with proof, whether τ is holomorphic or not.

Solution:

YOUR EXAM NUMBER ____
Please cross out this problem if you do not wish it graded

Problem 5A. *Score:*

(a) Suppose z, c_1, ..., c_n are distinct complex numbers, and

$$\frac{1}{z - c_1} + \cdots + \frac{1}{z - c_n} = 0.$$

Show that z lies in the convex hull of c_1, ..., c_n.
(b) Let $p(z)$ be a non-constant polynomial. Show that every zero of $p'(z)$ lies in the convex hull of the zeroes of $p(z)$.

Solution:

YOUR EXAM NUMBER ____

Please cross out this problem if you do not wish it graded

Problem 6A. *Score:*

In the Euclidean space $\mathbb{R}^4$, consider the "hyper-ellipsoid" $2x^2 + 3y^2 + 4z^2 + 5u^2 = 1$. Does there exist a 3-dimensional subspace passing through the origin which intersects the ellipsoid in a sphere?

Solution:

YOUR EXAM NUMBER ___

Please cross out this problem if you do not wish it graded

Problem 7A. *Score:*

It is a corollary to the Jordan canonical form theorem that $n \times n$ matrices in Jordan canonical form, all of whose eigenvalues are zeroes, are similar if and only if the sizes of their Jordan blocks coincide (up to permutations). Prove this directly, without using the Jordan canonical form theorem.

Solution:

YOUR EXAM NUMBER ____

Please cross out this problem if you do not wish it graded

Problem 8A. *Score:*

Find all the subgroups of the dihedral group of order 12 (the group of symmetries of a regular hexagon).

Solution:

YOUR EXAM NUMBER ___

Please cross out this problem if you do not wish it graded

Problem 9A. *Score:*

Show that $x^3 - 2x$ is an injective function from the rational numbers to the rational numbers.

Solution:

Department of Mathematics, University of California, Berkeley

YOUR 1 OR 2 DIGIT EXAM NUMBER ____

GRADUATE PRELIMINARY EXAMINATION, Part B — Fall Semester 2015

---

1. Please write your 1- or 2-digit exam number on this cover sheet and on **all** problem sheets (even problems that you do not wish to be graded).
2. Indicate below which six problems you wish to have graded. **Cross out** solutions you may have begun for the problems that you have not selected.
3. Extra sheets should be stapled to the appropriate problem at the upper right corner. Do not put work for problem p on either side of the page for problem q if $p \neq q$.
4. No notes, books, calculators or other electronic devices may be used during the exam.

PROBLEM SELECTION

Part B: List the six problems you have chosen:

______ , ______ , ______ , ______ , ______ , ______

YOUR EXAM NUMBER ____

Please cross out this problem if you do not wish it graded

Problem 1B. *Score:*

For a real, find a 2-dimensional space of real-valued solutions of $y'' = ay/x^2$ for $x > 0$. When $a = -1/4$ find the solution with $y = 0, y' = 1$ at $x = 1$.

Solution:

YOUR EXAM NUMBER ____

Please cross out this problem if you do not wish it graded

Problem 2B. *Score:*

Let $f\colon [0,\infty) \to \mathbb{R}$ be a function, and assume that:

- f is continuous on $[0,\infty)$;
- f is differentiable on $(0,\infty)$;
- $f'(x) \le 0$ for all $x > 0$ such that $f(x) > 1$; and
- $f(0) = 1$.

Prove that $f(x) \le 1$ for all $x \ge 0$.

Solution:

YOUR EXAM NUMBER ____

Please cross out this problem if you do not wish it graded

Problem 3B. *Score:*

The unit cube in the space $C[0,1]$ of continuous real-valued functions on the interval is defined as the subset

$$\{f \in C[0,1] \mid \|f\| := \sup_{0 \le t \le 1} |f(t)| \le 1\}.$$

Prove that there exists a 2-dimensional linear subspace in $C[0,1]$ whose intersection with the unit cube is a circular disk.

Solution:

YOUR EXAM NUMBER ____
Please cross out this problem if you do not wish it graded

Problem 4B. *Score:*

A Schur function is a non-constant holomorphic function defined in the open unit disk whose values have absolute value at most 1. Show that if f is a Schur function then

$$\frac{f(0) - f(z)}{(1 - \overline{f(0)}f(z))z}$$

is also a Schur function.

Solution:

YOUR EXAM NUMBER ___

Please cross out this problem if you do not wish it graded

Problem 5B. *Score:*

Find all entire functions $f(z)$ such that $\mathrm{Re}(f(x+iy)) = x^3y - x\,y^3$. Express your answer directly in terms of z, not in terms of x and y.

Solution:

YOUR EXAM NUMBER ____
Please cross out this problem if you do not wish it graded

Problem 6B. *Score:*

Given a positive integer n, let $\dots c_{-1}, c_0, c_1, \dots$ be a sequence of real numbers with period n, that is, $c_{k+n} = c_k$ for all $k \in \mathbf{Z}$. Let C be the $n \times n$-matrix defined by $c_{ij} = c_{j-i}$. Prove that all matrices of this form (for n fixed) have a common Hermitian-orthonormal basis of complex eigenvectors, find these eigenvectors, and the corresponding eigenvalues.

Solution:

YOUR EXAM NUMBER ____

Please cross out this problem if you do not wish it graded

Problem 7B. *Score:*

Find the number of surjective linear maps from an n-dimensional vector space over the field with 2 elements to itself.

Solution:

YOUR EXAM NUMBER ____

Please cross out this problem if you do not wish it graded

Problem 8B. *Score:*

If A is the ring of $n \times n$ matrices with entries in a field K, show that the only two-sided ideals of A are A itself and 0.

Solution:

YOUR EXAM NUMBER ___

Please cross out this problem if you do not wish it graded

Problem 9B. *Score:*

How many ways are there to arrange 8 rooks on an 8 by 8 chessboard so that no two attack each other (in other words, each row and column contains exactly one rook), where two ways are counted as the same if they are equivalent under one of the 8 symmetries of the chessboard? You may assume the Polya–Burnside theorem that the number of orbits of a finite group on a finite set is the average number of fixed points of elements of the group.

Solution: