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Topology Qualifying Exam

The Ph.D. qualifying exam committee tries to proofread the examinations as carefully as possible. Nevertheless, the exam may contain misprints. If you are convinced a problem has been stated incorrectly, mention this to the proctor and indicate your interpretation in your solution. In such cases do not interpret the problem in such a way that it becomes trivial.

Directions: Do four problems in each section. Budget your time. Write your solution for each question on a separate page.

Section I

1. Let A and B be closed subspaces of a topological space X with $X = A \bigcup B$. Suppose that $f : A \to Y$ and $g : B \to Y$ are continuous, and $f(x) = g(x)$ for all $x \in A \bigcap B$.

   Prove that $h : X \to Y$ by

   $$h(x) = \begin{cases} f(x) & \text{if} \quad x \in A \\ g(x) & \text{if} \quad x \in B \end{cases}$$

   is continuous. Is it necessary for both A and B to be closed? Discuss.

2. Let $f : X \to Y$ by a quotient map. Let Y be connected and suppose that for each $y \in Y$, $f^{-1}(y)$ is connected. Prove that X is connected.

3. Let I be a non empty index set, let $\{X_\alpha | \alpha \in I\}$ be a family of topological spaces, and let $A_\alpha \subseteq X_\alpha$ for each α.

   (a) Show that if A_α is closed in X_α for each α, then $\prod A_\alpha$ is closed in $\prod X_\alpha$.

   (b) Show that $\overline{\prod A_\alpha} = \prod \overline{A_\alpha}$.

   (c) Prove or disprove: If A_α is open in X_α for each α, then $\prod A_\alpha$ is open in $\prod X_\alpha$.

4. Let D be the closed unit disk in the complex plane. Let $\sim$ be the equivalence relation on D defined by $z_1 \sim z_2$ if and only if $z_1 = z_2$ or $|z_1| = |z_2| < 1$. Is the quotient topological space Hausdorff? (Prove your assertion.)

5. State the definition of compactness for topological spaces. Prove <u>from your definition</u> that the closed unit interval $[0, 1]$ is compact.

Section II

1. Define what it means for Y to be a strong deformation retract of X, where $Y \subseteq X$ are topological spaces. Prove that if $i : Y \to X$ is the inclusion map and $y \in Y$, then the induced homomorphism $i_* : \pi_1(Y, y) \to \pi_1(X, y)$ is an isomorphism.

2. Prove by any method you know that:

   (a) $\Re$ is not homeomorphic to $\Re^2$.

   (b) $\Re^2$ is not homeomorphic to $\Re^3$.

3. Let X_1 and X_2 be two copies of S^2 and let N_1, S_1 and N_2, S_2 be the north and south poles of X_1 and X_2, respectively. Define X to be the quotient space obtained by identifying N_1 with N_2 and S_1 with S_2. Compute the fundamental group of X by using the Seifert-van Kampen theorem.

4. (a) Define a covering space.

   (b) State the main theorem about path lifting and covering spaces.

   (c) Let $S^1 \bigvee \Re P^2$ be the one point union of the circle and two dimensional real projective space, *i.e.* the quotient space obtained by taking the disjoint union of S^2 and $\Re P^2$ and then identifying a single point $x \in S^2$ with a single point in $y \in \Re P^2$. Describe the universal cover of $S^1 \bigvee \Re P^2$.

   (d) Describe the fundamental group of $S^1 \bigvee \Re P^2$.

5. Prove that $\Re^2$ cannot be retracted to S^1.