Preliminary Exam - Fall 1978

Problem 1 *Let $f:\mathbb{R}\to\mathbb{R}$ satisfy $f(x)\leqslant f(y)$ for $x\leqslant y$. Prove that the set where f is not continuous is finite or countably infinite.*

Problem 2 *Let $\{g_n\}$ be a sequence of Riemann integrable functions from $[0,1]$ into $\mathbb{R}$ such that $|g_n(x)|\leqslant 1$ for all n, x. Define*

$$G_n(x)=\int_0^x g_n(t)\,dt\,.$$

Prove that a subsequence of $\{G_n\}$ converges uniformly.

Problem 3 *Let $M_{n\times n}$ denote the vector space of $n\times n$ real matrices. Prove that there are neighborhoods U and V in $M_{n\times n}$ of the identity matrix such that for every A in U, there is a unique X in V such that $X^4=A$.*

Problem 4 *Evaluate*

$$\int_0^{2\pi}\frac{d\theta}{1-2r\cos\theta+r^2}$$

where $r^2\neq 1$.

Problem 5 *Let $f(z)=a_0+a_1z+\cdots+a_nz^n$ be a complex polynomial of degree $n>0$. Prove*

$$\frac{1}{2\pi i}\int_{|z|=R} z^{n-1}|f(z)|^2\,dz=a_0\overline{a}_nR^{2n}.$$

Problem 6 *Solve the differential equation*

$$\frac{dy}{dx}=x^2y-3x^2,\qquad y(0)=1.$$

Problem 7 *Let H be a subgroup of a finite group G.*

1. *Show that H has the same number of left cosets as right cosets.*
2. *Let G be the group of symmetries of the square. Find a subgroup H such that $xH\neq Hx$ for some x.*

Problem 8 *Let M be the $n \times n$ matrix over a field $\mathbf{F}$, all of whose entries are equal to 1.*

1. *Find the characteristic polynomial of M.*
2. *Is M diagonalizable?*
3. *Find the Jordan Canonical Form of M and discuss the extent to which the Jordan form depends on the characteristic of the field $\mathbf{F}$.*

Problem 9 *For $x, y \in \mathbb{C}^n$, let $\langle x, y\rangle$ be the Hermitian inner product $\sum_j x_j \overline{y}_j$. Let T be a linear operator on $\mathbb{C}^n$ such that $\langle Tx, Ty\rangle = 0$ if $\langle x, y\rangle = 0$. Prove that $T = kS$ for some scalar k and some operator S which is unitary: $\langle Sx, Sy\rangle = \langle x, y\rangle$ for all x and y.*

Problem 10 *How many homomorphisms are there from the group$\mathbb{Z}_2 \times \mathbb{Z}_2$ to the symmetric group on three objects?*

Problem 11 *Let $W \subset \mathbb{R}^n$ be an open connected set and f a real valued function on W such that all partial derivatives of f are 0. Prove that f is constant.*

Problem 12 *Let $f : \mathbb{R}^n \to \mathbb{R}$ have the following properties: f is differentiable on $\mathbb{R}^n \setminus \{0\}$, f is continuous at 0, and*

$$\lim_{p\to 0} \frac{\partial f}{\partial x_i}(p) = 0$$

for $i = 1, \ldots, n$. Prove that f is differentiable at 0.

Problem 13

1. *Show that if $u, v : \mathbb{R}^2 \to \mathbb{R}$ are continuously differentiable and $\dfrac{\partial u}{\partial y} = \dfrac{\partial v}{\partial x}$, then $u = \dfrac{\partial f}{\partial x}$, $v = \dfrac{\partial f}{\partial y}$ for some $f : \mathbb{R}^2 \to \mathbb{R}$.*
2. *Prove there is no $f : \mathbb{R}^2 \setminus \{0\} \to \mathbb{R}$ such that*

$$\frac{\partial f}{\partial x} = \frac{-y}{x^2 + y^2} \quad \text{and} \quad \frac{\partial f}{\partial y} = \frac{x}{x^2 + y^2}.$$

Problem 14 *Give examples of conformal maps as follows:*

1. *from $\{z \mid |z| < 1\}$ onto $\{z \mid \Re z < 0\}$,*

2. *from* $\{z \mid |z| < 1\}$ *onto itself, with* $f(0) = 0$ *and* $f(1/2) = i/2$,

3. *from* $\{z \mid z \neq 0, 0 < \arg z < \frac{3\pi}{2}\}$ *onto* $\{z \mid z \neq 0, 0 < \arg z < \frac{\pi}{2}\}$.

Problem 15 *Suppose* $h(z)$ *is analytic in the whole plane,* $h(0) = 3+4i$, *and* $|h(z)| \leqslant 5$ *if* $|z| < 1$. *What is* $h'(0)$*?*

Problem 16 *Which of the following matrices are similar as matrices over* $\mathbb{R}$*?*

$$(a) \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad (b) \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad (c) \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$(d) \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}, \quad (e) \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad (f) \begin{pmatrix} 0 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

Problem 17 *Find all automorphisms of the additive group of rational numbers.*

Problem 18 *Prove that every finite multiplicative group of complex numbers is cyclic.*

Problem 19 *Let*

$$A = \begin{pmatrix} 1 & 2 \\ 1 & -1 \end{pmatrix}.$$

Express A^{-1} *as a polynomial in* A *with real coefficients.*

Problem 20 *Let* $M_{n\times n}$ *be the vector space of real* $n \times n$ *matrices, identified with* $\mathbb{R}^{n^2}$. *Let* $X \subset M_{n\times n}$ *be a compact set. Let* $S \subset \mathbb{C}$ *be the set of all numbers that are eigenvalues of at least one element of* X. *Prove that* S *is compact.*