Zack Garza

1a) Note that if $x \in C$ is an endpoint of a removed interval, then $x = k/3^n$ for some integers $n \geq 1$ and $0 \leq k \leq 3^n$. So we just need a real number $x \in (0,1)$ satisfying

a) x has <u>some</u> ternary expansion

$$x = \sum_{i=1}^{\infty} a_i 3^{-i} \text{ where } a_i \neq 1 \text{ for any } i, \text{ and}$$

b) $x \neq k/3^n$ for any $k, n \in \mathbb{N}^{>0}$,

then we will have $x \in C$ by (a) and x not an endpoint by (b).

<u>Claim</u>: $x = (0.\overline{02})_3 = (0.020202\cdots)_3$ works.

Base 3

<u>Pf</u>: By construction, x satisfies

(a) $$x = \sum_{i=0}^{\infty} a_i 3^{-i}, \quad a_i \in \{0, 2\}$$

So no $a_i = 1$ and thus $x \in C$.

(b) To see that x satisfies (b), we can compute

$$x = (0.020202\cdots)_3$$
$$= 0 \cdot 3^{-1} + 2 \cdot 3^{-2} + 0 \cdot 3^{-3} + 2 \cdot 3^{-4} + \cdots$$
$$= \sum_{i=1}^{\infty} 2 \cdot 3^{-2i} = 2 \sum_{i=1}^{\infty} 3^{-2i} = 2 \sum_{i=1}^{\infty} \left(\tfrac{1}{9}\right)^i$$
$$= 2\left(-1 + \sum_{i=0}^{\infty} \left(\tfrac{1}{9}\right)^i\right)$$
$$= 2\left(-1 + \frac{1}{1-\frac{1}{9}}\right) \boxed{= 1/4,}$$

where $4 \neq 3^n$ for any integer n. ▨

1b If a set X is <u>nowhere dense</u> in a topological space, it equivalently satisfies

$$(\overline{X})^\circ = \emptyset$$

(i.e., the interior of the closure is empty.)

It then suffices to show that

a) C is closed, so $\overline{C} = C$, and

b) C has no interior points, so $C^\circ = \emptyset$.

(a) To see that C is closed, we will show $C^c := [0,1] \setminus C$ is open. An arbitrary union of open sets is open, so the claim is that $C^c = \bigcup_{j \in J} A_j$ for some collection of open sets $\{A_j\}_{j \in J}$.

Consider C_n, the n^{th} stage of the process used to construct the Cantor set, so $C = \bigcap_{i=1}^{\infty} C_n$.

But by induction, C_n^c is a union of open sets.

In particular, $C_1^c = (\frac{1}{3}, \frac{2}{3})$, and

$$C_n^c = \underbrace{\left(\bigcup_{i=1}^{n-1} C_i^c \right)}_{\text{Open by hypothesis}} \cup \underbrace{\left(\begin{array}{c} \text{Exactly } n \text{ open intervals} \\ \text{that were deleted} \end{array} \right)}_{\text{open by construction}},$$

So C_n^c is open for each n. But then

$$C^c = \left(\bigcap_{n=1}^{\infty} C_n\right)^c = \bigcup_{i=1}^{\infty} C_n^c$$

is a union of open sets and thus open. So C is closed.

(b) To see that $C^\circ = \emptyset$, suppose towards a contradiction that $x \in C^\circ$, so there exists some $\varepsilon > 0$ such that $N_\varepsilon(x) := (x-\varepsilon, x+\varepsilon) \subsetneq C$. Letting $\mu(I)$ denote the length of an interval, we have $\mu(N_\varepsilon(x)) = 2\varepsilon > 0$.

<u>Claim</u>: Let $L_n := \mu(C_n)$, then $L_n = \left(\frac{2}{3}\right)^n$.

This follows immediately by noting that L_n satisfies the recurrence relation

$$L_{n+1} = \frac{2}{3} L_n, \qquad L_0 = 1$$

Since an interval of length $\frac{1}{3} L_{n-1}$ is removed at the n^{th} stage, which has the unique claimed solution.

But if $I_1 \subseteq I_2$ are real intervals, we must have $\underbrace{\mu(I_1) \leq \mu(I_2)}_{\text{Uses subadditivity of measure}}$, whereas if we choose n large enough such that $\left(\frac{2}{3}\right)^n < 2\varepsilon$, we have

$$(x-\varepsilon, x+\varepsilon) \subsetneq C = \bigcap_{i=1}^{\infty} C_i \Rightarrow (x-\varepsilon, x+\varepsilon) \subseteq C_n, \text{ but}$$

$$\mu((x-\varepsilon, x+\varepsilon)) = 2\varepsilon > \left(\frac{2}{3}\right)^n = \mu(C_n), \text{ a contradiction.}$$

So such an $x \in C^\circ$ can't exist, and $C^\circ = \emptyset$.

Thus $(\overline{C})^\circ = C^\circ = \emptyset$, and C is nowhere dense, and since a meager set is a countable union of nowhere dense sets, C is meager. $\square$

<u>Claim</u>: C is measure zero.

Measures are additive over disjoint sets, i.e.

$$A \cap B = \emptyset \Rightarrow \mu(A \sqcup B) = \mu(A) + \mu(B),$$

And if $A \subseteq B$, we have

$$\mu(B) = \mu(B \sqcup (B \setminus A)) = \mu(B) + \mu(B \setminus A)$$
$$\Rightarrow \mu(B \setminus A) = \mu(B) - \mu(A).$$

Now let B_n be the union of the intervals that are deleted at the n^{th} step. We have

$$\mu(B_0) = 0$$

$$\mu(B_1) = 1/3$$

$$\mu(B_2) = 2\left(\tfrac{1}{9}\right) = 2/9$$

$$\mu(B_3) = 4\left(\tfrac{1}{27}\right) = 4/27$$

$$\vdots$$

$$\mu(B_n) = 2^{n-1}/3^n$$

Moreover, if $i \neq j$, then $B_i \cap B_j = \emptyset$, and

$$C^c := [0,1] - C = \bigsqcup_{i=1}^{\infty} B_i .$$

We thus have

$$\begin{aligned}
\mu(C) &= \mu([0,1]) - \mu(C^c) \\
&= 1 - \mu\left(\bigsqcup_{n=1}^{\infty} B_n\right) \\
&= 1 - \sum_{n=1}^{\infty} \mu(B_n) \\
&= 1 - \sum_{n=1}^{\infty} 2^{n-1}/3^n
\end{aligned}$$

$$= 1 - (1/3)\sum_{n=0}^{\infty}\left(\tfrac{2}{3}\right)^n$$

$$= 1 - (1/3)\left(\frac{1}{1-2/3}\right)$$

$$= 0. \qquad \blacksquare$$

1c Let $y \in [0,1]$ be arbitrary, we will produce an $x \in C$ such that $f(x) = c$.

Write $y = (a_1 a_2 \cdots)_2 = \sum_{i=1}^{\infty} a_i 2^{-i}$ where $a_i \in \{0,1\}$

Now define

$$x = (2a_1 2a_2 \cdots)_3 = \sum_{i=1}^{\infty} (2a_i)3^{-i} := \sum_{i=1}^{\infty} b_i 3^{-i}$$

Since $a_i \in \{0,1\}$, $b_i = 2a_i \in \{0,2\}$, meaning x has no 1^s in its ternary expansion and so $x \in C$.

Moreover, under f we have

$$\left.\begin{array}{ccc} b_i & \mapsto & \frac{1}{2} b_i \\ \| & & \| \\ 2a_i & \mapsto & \frac{1}{2}(2a_i) = a_i \end{array}\right\} \text{So } b_i \mapsto a_i \text{ and thus } f(x) = y.$$

So $C \twoheadrightarrow [0,1]$, which is uncountable, thus so is C. $\blacksquare$

(2a) $(\Rightarrow)$ Suppose X is G_δ, so $X = \bigcup_{n=1}^{\infty} A_i$ with each A_i closed. Then A_i^c is open by definition, and so

$$X^c = \left(\bigcup_{i=1}^{\infty} A_i\right)^c = \bigcap_{i=1}^{\infty} A_i^c$$

is a countable intersection of open sets, and thus F_σ.

$(\Leftarrow)$ Suppose X^c is an F_σ, so $X^c = \bigcap_{i=1}^{\infty} B_i$ with each B_i open. Then each B_i^c is closed by definition, and

$$X = (X^c)^c = \left(\bigcap_{i=1}^{\infty} B_i\right)^c = \bigcup_{i=1}^{\infty} B_i^c$$

is a countable union of closed sets, and thus G_δ.

(2b) Suppose X is closed, we will show $X = \bigcap_{n=1}^{\infty} C_n$ with each C_n open. For each $x \in X$ and $n \in \mathbb{N}$, define

- $B_n(x) = \{ y \in \mathbb{R}^n \mid d(x,y) < \frac{1}{n} \}$
- $C_n = \bigcup_{x \in X} B_n(x)$
- $W = \bigcap_{n=1}^{\infty} C_n = \bigcap_{n=1}^{\infty} \bigcup_{x \in X} B_n(x)$

Since each $B_n(x)$ is open by construction and C_n is a union of opens, each C_n is open.

Claim: $W = X$.

$X \subseteq W$: If $x \in X$, then $x \in B_n(x) \subseteq C_n$ for all n, and so

$$x \in \bigcap_{n=1}^{\infty} C_n = W.$$

$W \subseteq X$: Suppose there is some $w \in W \setminus X$ (so $w \neq x$ for any $x \in X$) towards a contradiction.

Since $w \in \bigcap_{i=1}^{n} C_n$, $w \in C_n$ for every n. So $w \in \bigcup_{x \in X} B_n(x)$ for every n. But then there is some particular $x_0 \in X$ such that $w \in B_n(x_0)$ for every n (otherwise we could take N large enough so that $w \notin B_N(x)$ for any $x \in X$, so $x \notin \bigcup_{x \in X} B_N(x)$) where $w \neq x_0$.

But then if $N_\varepsilon(x)$ is an arbitrary neighborhood of x, we can take $\frac{1}{n} < \varepsilon$ to obtain $w \in B_n(x) \subsetneq N_\varepsilon(x)$, which makes w a limit point of X. But since X is closed, it contains its limit points, forcing the contradiction $w \in X$.

So X is a countable intersection of open sets, and thus a G_δ set. $\blacksquare$

Now suppose X is open. Then X^c is closed, and thus a G_δ set. But then $(X^c)^c = X$ is an F_σ set by problem (2a). $\blacksquare$

(2c) Using the fact that singletons are closed in metric spaces, we can write $\mathbb{Q} = \bigcup_{q \in \mathbb{Q}} \{q\}$ as a countable union of closed sets, so $\mathbb{Q}$ is an F_δ set. Suppose $\mathbb{Q}$ was also a G_δ set, so $\mathbb{Q} = \bigcap_{i=1}^{\infty} A_i$ with each A_i open. Then for any fixed n, $\mathbb{Q} \subseteq A_i$, so A_i is dense in $\mathbb{R}$ for every i.

However, it is also true that $\{q\}^c := \mathbb{R} \setminus \{q\}$ is an open, dense subset of $\mathbb{R}$, and we can write

$$\mathbb{R} \setminus \mathbb{Q} = \mathbb{R} \setminus \bigcup_{q \in \mathbb{Q}} \{q\} = \bigcap_{q \in \mathbb{Q}} (\mathbb{R} \setminus \{q\})$$

as in intersection of open dense sets; since $\mathbb{R}$ is a Baire space, countable intersections of open dense sets are dense.

But then $\left(\bigcap_{i=1}^{\infty} A_i\right) \cap \left(\bigcap_{q \in \mathbb{Q}} \{q\}^c\right) = \mathbb{Q} \cap (\mathbb{R} \setminus \mathbb{Q}) = \varnothing$ must be dense in $\mathbb{R}$, which is absurd. ※

Note that this argument also works when $\mathbb{R}$ is replaced with any open interval I and $\mathbb{Q}$ is replaced with $\mathbb{Q} \cap I$.

For a set that is neither G_δ nor F_δ, consider

$A = \mathbb{Q} \cap (0, \infty)$, positive rationals

$B = (\mathbb{R} \setminus \mathbb{Q}) \cap (-\infty, 0)$, negative irrationals

A is F_σ but not G_δ , using above argument, and

dually B is G_δ but not F_σ.

<u>Claim</u>: $X = A \cup B$ is neither G_δ nor F_σ.

Suppose X is G_δ. Then $X \cap \overbrace{(0,\infty)}^{\text{open}} = A$ is G_δ as well. #

Suppose X is F_σ. Then X^c is G_δ, but

$$X^c = (A \cup B)^c = A^c \cap B^c = (\mathbb{Q} \cap (-\infty, 0)) \cup ((\mathbb{R} \setminus \mathbb{Q}) \cap (0, \infty))$$

and thus $X^c \cap \overbrace{(-\infty, 0)}^{\text{open}} = A$ is G_δ. #

So X is neither G_δ or F_σ.

$\blacksquare$

(3a) <u>Claim:</u> $c \in [0,1] \Rightarrow \lim_{x\to c} f(x) = 0.$

This holds iff $\forall c \in I, \forall \varepsilon, \exists \delta$ s.t. $|x-c| < \delta \Rightarrow |f(x)| < \varepsilon,$

so let $\varepsilon > 0$ be arbitrary. Consider the set

$S = \{ n \in \mathbb{N} \mid \frac{1}{n} \geq \varepsilon \}$, which is a <u>finite</u> set, and so

$S_q = \{ r_n \in \mathbb{Q} \mid \frac{1}{n} \geq \varepsilon \}$ is <u>finite</u> as well.

So choose $\delta < \min_{r_n \in S_q} d(c, r_n)$ so $N_\delta(c) \cap S_q = \emptyset$

Then $|x-c| < \delta \Rightarrow \begin{cases} \cdot\, f(x) = 0 \text{ if } x \in I \setminus \mathbb{Q}, \text{ or} \\ \cdot\, x = r_m \in (\mathbb{Q} \setminus S_q) \cap I \text{ for some } m \text{ such that} \\ \quad 1/m < \varepsilon \text{ by construction.} \end{cases}$

But then $|f(x)| = |1/m| < \varepsilon$ as desired. $\square$

So $\cdot\; c \in I \setminus \mathbb{Q} \Rightarrow f(c) = 0 = \lim_{x\to c} f(x),$

$\cdot\; c = r_n \in I \cap \mathbb{Q} \Rightarrow f(c) = \frac{1}{n} \neq 0 = \lim_{x\to c} f(x)$

and f is discontinuous on $I \cap \mathbb{Q}$. $\blacksquare$

(3b.1) Claim: ω_f is well-defined

This amounts to showing that the sup and limit exist in

$$\omega_f(x) = \lim_{\delta \to 0^+} \sup_{y,z \in B_\delta(x)} |f(y) - f(z)|$$

Let $x \in \mathbb{R}$ be arbitrary and δ fixed.

Since f is bounded, there is some M such that

$\forall y \in \mathbb{R}$, $|f(y)| < M$, and so

$$y,z \in \mathbb{R} \Rightarrow |f(y) - f(z)| = |f(y) + (-f(z))| \leq |f(y)| + |-f(z)|$$
$$= |f(y)| + |f(z)| < 2M,$$

which holds for $y,z \in B_\delta(x) \subseteq \mathbb{R}$ as well.

And so $\{ |f(y) - f(z)| \text{ s.t. } y,z \in B_\delta(x) \}$ is bounded above and thus has a least upper bound, and thus the following supremum exists.

$$S(\delta, x) = \sup_{y,z \in B_\delta(x)} |f(y) - f(z)|$$

To see that the $\lim_{\delta \to 0} S(\delta, x)$ exists, note that

$$\delta_1 \leq \delta_2 \Rightarrow B_{\delta_1}(x) \subseteq B_{\delta_2}(x)$$

and so for a fixed x, $S(\delta, x)$ is a monotonically decreasing function of δ that is bounded below by 0,

which converges by the monotone convergence theorem. $\square$

Claim: f is continuous at x iff $\omega_f(x) = 0$.

($\Leftarrow$) Suppose $\omega_f(x) = 0$ and let $\varepsilon > 0$ be arbitrary; we will produce a δ to use in the definition of continuity.

Since $\omega_f(x) = \lim_{d \to 0^+} S(d, x) = 0$, we can choose δ such that

$$d < \delta \Rightarrow |S(d,x)| < \varepsilon \quad , \qquad \text{which means}$$

$$d < \delta \Rightarrow \sup_{y,z \in B_d(x)} |f(y) - f(z)| < \varepsilon$$

So fix $z = x$ and let y vary, yielding

$$d < \delta \Rightarrow \sup_{y \in B_d(x)} |f(y) - f(x)| < \varepsilon$$

But now for an arbitrary $t \in B_\delta(x)$, we have $|x-t| < \delta$ and

$$|f(x) - f(t)| \leq \sup_{y \in B_\delta(x)} |f(x) - f(y)| < \varepsilon,$$

which exactly says $|x-t| < \delta \Rightarrow |f(x) - f(t)| < \varepsilon$. $\square$

($\Rightarrow$) Suppose f is continuous at x and let $\varepsilon > 0$ be arbitrary; we will show $\omega_f(x) < \varepsilon$.

Since f is continuous, choose δ such that

$$|x-y| < \delta \Rightarrow |f(x) - f(y)| < \varepsilon/2.$$

We then have

$$y, z \in B_\delta(x) \Rightarrow |x-y| < \delta \quad \text{and} \quad |x-z| < \delta,$$

$$\Rightarrow |f(x) - f(y)| < \frac{\varepsilon}{2} \quad \text{and} \quad |f(x) - f(z)| < \frac{\varepsilon}{2}$$

$$\Rightarrow |f(y) - f(z)| \leq |f(y) - f(x)| + |f(x) - f(z)| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon,$$

and so

$$y, z \in B_\delta(x) \Rightarrow |f(y) - f(z)| < \varepsilon \quad \Rightarrow \sup_{y,z \in B_\delta(x)} |f(y) - f(z)| \leq \varepsilon$$

$$\Rightarrow S(\delta, x) \leq \varepsilon,$$

and since $S(d,x)$ is monotonically decreasing in d,

$$\omega_f(x) = \lim_{d \to 0^+} S(d,x) \leq S(\delta, x) \leq \varepsilon$$

as desired. $\blacksquare$

(3b.2) We will show that

$$A_\varepsilon^c = \{ x \in \mathbb{R} \mid \omega_f(x) < \varepsilon \}$$

is open by showing every point is an interior point.

Fix $\varepsilon > 0$ and let $x \in A_\varepsilon^c$ be arbitrary. We want to produce a δ such that

$$B_\delta(x) \subsetneq A_\varepsilon^c \quad \text{or equivalently} \quad |y-x| < \delta \Rightarrow \omega_f(y) < \varepsilon.$$

Write $\omega_f(x) = \lim_{d \to 0^+} S(d,x)$; since $\omega_f(x) < \varepsilon$ and this limit exists, we can choose δ such that

$$d < \delta \Rightarrow |S(d,x) - 0| < \varepsilon \Rightarrow |S(d,x)| < \varepsilon.$$

Now suppose $y \in B_\delta(x)$, so $|y-x| < \delta$. Then there exists some δ' such that $B_{\delta'}(y) \subset B_\delta(x)$, and we claim that

$$S(\delta', y) \leq S(\delta, x)$$

Note that if this is true, then

S is monotonically decreasing in d

$$\omega_f(y) = \lim_{d \to 0} S(d,y) \leq S(\delta', y) \leq S(\delta, x) < \varepsilon.$$

To see why this is true, we just note that

$$a,b \in B_{\delta'}(y) \subset B_\delta(x) \Rightarrow a,b \in B_\delta(x)$$

$$\Rightarrow \sup_{a,b \in B_{\delta'}(y)} |f(y)-f(z)| \leq \sup_{y,z \in B_\delta(x)} |f(y)-f(z)|,$$

since the supremum can only increase over a larger set.

So $\omega_f(y) < \varepsilon$ as desired. $\blacksquare$

Finally, note that if $D_f = \{x \in \mathbb{R} \mid f \text{ is discontinuous at } x\}$,

then
$$D_f = \{x \in \mathbb{R} \mid \omega_f(x) \neq 0\} = \bigcup_{n=1}^{\infty} \{x \in \mathbb{R} \mid \omega_f(x) \geq \tfrac{1}{n}\}$$
$$= \bigcup_{n=1}^{\infty} A_{\frac{1}{n}}$$

is a countable union of closed sets and thus F_σ. $\blacksquare$

(4) <u>Claim</u>: f is increasing, i.e. $x \le y \Rightarrow f(x) \le f(y)$

Fix $x \in \mathbb{R}$, and define

$$A_x := \{ t \in X \mid x > t \}, \quad A_x^c := \{ t \in X \mid x \le t \}.$$

(Note that $t \in A_x$ or $t \in A_x^c \Rightarrow t = x_n$ for some n, and $X = A_x \sqcup A_x^c$.)

Then noting that

$$x_n \in A_x \Rightarrow f_n(x) \equiv 1$$

and

$$x_n \in A_x^c \Rightarrow f_n(x) \equiv 0,$$

We can write

$$f(x) = \sum_{n=1}^{\infty} \frac{1}{n^2} f_n(x) = \sum_{\{n \mid x_n \in A_x\}} \frac{1}{n^2} \cdot 1 + \sum_{\{n \mid x_n \in A_x^c\}} \frac{1}{n^2} \cdot 0$$

$$= \sum_{\{n \mid x_n \in A_x\}} \frac{1}{n^2}.$$

Now if $y \ge x$, then $y \ge t$ for every $t \in A_x$, so $A_y \supseteq A_x$.

But then

$$f(x) = \sum_{\{n \mid x_n \in A_x\}} \frac{1}{n^2} \leq \sum_{\{n \mid x_n \in A_y\}} \frac{1}{n^2} = f(y),$$

where the inequality holds because

$$A_x \subseteq A_y \Rightarrow \{n \mid x_n \in A_x\} \subseteq \{n \mid x_n \in A_y\}$$
$$\Rightarrow |\{n \mid x_n \in A_x\}| \leq |\{n \mid x_n \in A_y\}|,$$

so the latter sum has at least as many terms and everything is positive. So $f(x) \leq f(y)$.

Claim: f is continuous on $\mathbb{R} \setminus X$ since $\sum f_n \xrightarrow{u} f$ and each f_n is continuous there.

Since $|f_n(x)| \leq 1$ by definition, and $|f_n(x)/n^2| \leq |1/n^2| := M_n$ where $\sum M_n < \infty$, $\sum f_n \xrightarrow{u} f$ by the M test.

Note that for a fixed n, $D_{f_n} = \{x_n\}$. This is because if we take a sequence $\{y_i\} \to x_n$ with each $y_i > x_n$, then $f(y_i) = 1$ for every i, and

$$\lim_{i\to\infty} f(y_i) = \lim_{i\to\infty} 1 = 1 \neq f(\lim_{i\to\infty} y_i) = f(x_n) = 0$$

So f_n is not continuous at $x = x_n$. Otherwise, either $x > x_n$ or $x < x_n$, in which case we can let ε be arbitrary and choose $\delta < |x - x_n|$ to get

$$y \in B_\delta(x) \Rightarrow \begin{cases} y > x_n \Rightarrow |f(y) - f(x)| = |0-0| < \varepsilon \\ y < x_n \Rightarrow |f(y) - f(x)| = |1-1| < \varepsilon. \end{cases}$$

Letting $F_N = \sum_{n=1}^{N} f_n$, we find that

$$F_N = \underset{\uparrow}{f_1} + \underset{\uparrow}{f_2} + \ldots + \underset{\uparrow}{f_N}$$

discontinuous at: $\{x_1\} \cup \{x_2\} \cup \cdots \cup \{x_N\}$

$\Bigg\}$ So F_N is continuous on $\mathbb{R} \setminus \bigcup_{i=1}^{N} \{x_N\}$.

and since $\mathbb{R} \setminus X \subseteq \mathbb{R} \setminus \bigcup_{i=1}^{N} \{x_N\}$, F_N is continuous there too.

But then $f = \text{uniform limit}(F_N)$ is continuous on $\mathbb{R} \setminus X$. ∎

5a Let $X = (C(I), \|\cdot\|_\infty)$ where $I = [0,1]$,

$C(I) = \{f : I \to \mathbb{R} \mid f \text{ is continuous}\}$, and

$$d(f,g) = \|f-g\|_\infty = \sup_{x\in I} |f(x) - g(x)| .$$

Claim: X is a metric space.

1) $d(f,g) = 0 \Rightarrow f = g$

If $\sup_{x \in I} |f(x)-g(x)| = 0$ then $|f(x)-g(x)| = 0 \ \forall x \in \mathbb{R}$,

so $f(x) = g(x) \ \forall x \in \mathbb{R}$ and $f = g$.

2) $d(f,g) = d(g,f)$

We have
$$\begin{aligned} d(f,g) &= \sup_{x\in I} |f(x)-g(x)| \\ &\sup_{x\in I} |g(x)-f(x)| \\ &= d(g,f). \end{aligned}$$

3) $d(f,h) \leq d(f,g) + d(g,h)$

We have
$$\begin{aligned} d(f,g) &= \sup_{x\in I} |f(x) - g(x)| \\ &= \sup_{x\in I} |f(x) - h(x) + h(x) - g(x)| \end{aligned}$$

$$\le \sup_{x\in I}\Big(|f(x)-h(x)|+|h(x)-g(x)|\Big)$$ ← Δ-ineq in $\mathbb{R}$

$$= \sup_{x\in I}|f(x)-h(x)| + \sup_{x\in I}|h(x)-g(x)|$$

$$= d(f,h)+d(h,g).$$

So X is a metric space. □

<u>Claim</u>: X is complete.

Let $\{f_i\}$ be a Cauchy sequence in X, we will show that it converges in X. Since $\{f_i\}$ is Cauchy in X, we have

$$\forall \varepsilon>0,\ \exists N_0 \mid n\ge m\ge N_0 \Rightarrow \|f_n-f_m\|_\infty<\varepsilon$$

First we will define a candidate limit function f, then show $f\in X$.

1) Define $f:=\lim_{n\to\infty} f_n$ by $f(x)=\lim_{n\to\infty} f_n(x)$.

This is well-defined; let $S_x=\{f_i(x)\}\subseteq \mathbb{R}$ for a fixed x, and we claim S_x is Cauchy in $\mathbb{R}$, which is complete.

This follows because if $\{f_i\}$ is Cauchy in X, then

$$|f_n(x)-f_m(x)|\le \sup_{x\in I}|f_n(x)-f_m(x)| = \|f_n-f_m\|_\infty \to 0.$$

2) $f \in X$, for which it suffices to show f is continuous.

Let $\varepsilon > 0$, and since $\{f_i\}$ is Cauchy, choose N_0 large s.t.

$$n \geq N_0 \Rightarrow \|f_n - f\|_\infty < \frac{\varepsilon}{3}.$$

Now fix $n \geq N_0$; since <u>f_n is continuous,</u> choose δ such that

$$|x-y| < \delta \Rightarrow |f_n(x) - f_n(y)| < \frac{\varepsilon}{3}$$

Then

$$\begin{aligned}
|x-y| < \delta \Rightarrow |f(x) - f(y)| &= |f(x) - f_n(x) + f_n(x) - f_n(y) + f_n(y) - f(y)| \\
&\leq |f(x) - f_n(x)| + |f_n(x) - f_n(y)| + |f_n(y) - f(y)| \\
&\leq \sup_{x \in I} |f(x) - f_n(x)| + |f_n(x) - f_n(y)| + \sup_{y \in I} |f_n(y) - f(y)| \\
&= \|f - f_n\|_\infty + |f_n(x) - f_n(y)| + \|f_n - f\|_\infty \\
&\leq \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon,
\end{aligned}$$

So f is continuous, $f = \lim f_n \in X$, and X is complete. ∎

(5b) Let $B = \{ f \in X \mid \|f\|_\infty \leq 1 \}$

<u>Claim:</u> B is closed.

Let f be a limit point of B, so there is some sequence $f_n \to f$ in X with each $f_n \in B$ so $\|f_n\|_\infty \leq 1 \; \forall n$.

Let $\varepsilon > 0$, and since $f_n \to f$ in X, choose N_0 such that

$$n \geq N_0 \Rightarrow \|f_n - f\| < \varepsilon$$

Then,

$$\begin{aligned} \|f\|_\infty &= \|f - f_n + f_n\|_\infty \\ &\leq \|f - f_n\|_\infty + \|f_n\|_\infty \\ &< \varepsilon + 1, \end{aligned}$$

and taking $\varepsilon \to 0$ yields $\|f\|_\infty \leq 1$. ∎

<u>Claim:</u> B is bounded

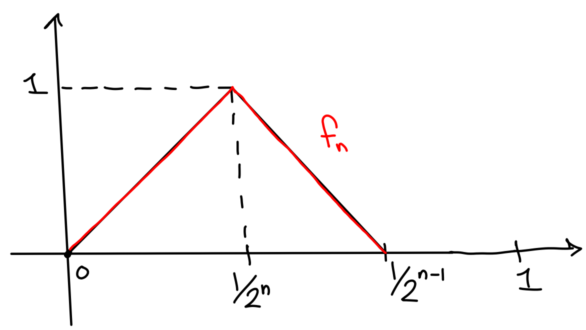
A subset $B \subseteq X$ is bounded iff there is some $x \in X$ and some $r > 0$ in $\mathbb{R}$ where $B \subset N(r, x) = \{ y \in X \mid d(y, x) < r \}$.

Choose $x = 0$, $r = 2$, then $f \in B \Rightarrow d(f, 0) = \|f - 0\|_\infty = 1 < 2$, so $f \in N(2, 0)$.

Claim: B is not compact.

Since B is a metric space, B is compact iff B is sequentially compact.

Define f_n as the triangle:



Then $f_n \xrightarrow{\mathbb{R}} f$ where $f(x) = \begin{cases} 1, & x=0 \\ 0, & x \in (0,1] \end{cases}$,

pointwise in $\mathbb{R}$

and so $\forall n$, $\|f_n - f\|_\infty = 1$, attained at $x=0$. So $\lim_{n\to\infty} \|f_n - f\|_\infty \neq 0$,

and $\{f_n\}$ does not converge in X, nor can any subsequence. ∎

Claim: B is not totally bounded.

If it were, $\forall \varepsilon$ there would exist a finite collection $\{g_i\}_{i=1}^N \subseteq B$ such that $B \subseteq \bigcup_{i=1}^N N(\varepsilon, g_i)$ where

$$N(\varepsilon, g_i) = \{h \in B \mid \|h - g_i\| < \varepsilon\}.$$

Note that if $h_1, h_2 \in N(\varepsilon, g_i)$ then $\|h_1 - h_2\| \leq \|h_1 - g\| + \|g - h_2\| < 2\varepsilon$.

So choose $\varepsilon = \frac{1}{2}$, and consider the collection $\{f_n\}_{n=1}^{\infty}$.

Since $\|f_n - f_m\| = 1$, each $N(\varepsilon, g_i)$ can contain at most one f_n, since $f_n, f_m \in N(\varepsilon, g_i)$ for $n \neq m$ would imply $\|f_n - f_m\|_\infty < 2\varepsilon = 2(\frac{1}{2}) = 1$. But there are finitely many $N(\varepsilon, g_i)$ and infinitely many f_n, so if this is a cover of B, so $N(\varepsilon, g_i)$ must contain at least 2 f_n's. ※

(6a) Claim: If $\sum g_n \xrightarrow{u} G$, then $g_n \xrightarrow{u} 0$.

Let $G_N = \sum_{n=1}^{N} g_n$ and $G = \lim_{N \to \infty} G_N$.

Suppose $G_N \xrightarrow{u} G$, then choose N large enough so that

$$\forall x \in X,\ n \geq N \Rightarrow |G_n(x) - G(x)| < \frac{\varepsilon}{2}$$

Then letting $n > n-1 > N$, we have

$$\begin{aligned}
|g_n(x)| &= \left| \sum_{i=1}^{n} g_i(x) - \sum_{i=1}^{n-1} g_i(x) \right| \\
&= \left| \left(\sum_{i=1}^{n} g_i(x) - G(x) \right) - \left(\sum_{i=1}^{n-1} g_i - G(x) \right) \right| \\
&\leq \left| \sum_{i=1}^{n} g_i(x) - G(x) \right| + \left| \sum_{i=1}^{n-1} g_i - G(x) \right| \\
&\leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon,
\end{aligned}$$

So $\forall x \in X$, $|g_n(x)| < \varepsilon \Rightarrow g_n \xrightarrow{u} 0$. $\square$

Now let $g_n = 1/1+n^2x$, we'll show g_n does <u>not</u> converge to 0 uniformly.

Note $g_n \xrightarrow{u} g$ iff $\forall \varepsilon, \exists N_0 \mid \forall x, n \geq N_0 \Rightarrow |g_n(x) - g(x)| < \varepsilon$,

so let $\varepsilon < \frac{1}{2}$, N_0 be arbitrary, and choose $x_0 < 1/N_0^2$. Then,

$$|g_{N_0}(x_0)| = \frac{1}{|1+N_0^2x|} = \frac{1}{|1+N_0^2(1/N_0^2)} = \frac{1}{2} > \varepsilon .$$

<u>Claim</u>: g is continuous on $(0, \infty)$.

Let $x \in (0,\infty)$ be arbitrary, and choose $a < x$. We will show g converges uniformly on $[a, \infty)$, and since each g_n is continuous on $[a, \infty)$ as well, g will be the uniform limit of continuous functions and thus continuous itself.

We can use the M-test. Since $x > a$,

$$\left|1/1+n^2x\right| \leq \left|1/n^2x\right| \leq \left|1/n^2a\right| = \frac{1}{a}\left|\frac{1}{n^2}\right|,$$

$$\text{where } \sum_{n=1}^{\infty} \frac{1}{a}\frac{1}{n^2} = \frac{1}{a}\sum \frac{1}{n^2} < \infty,$$

So g converges uniformly on $[a, \infty)$.

(6b) Claim: g is differentiable on $(0,\infty)$.

If $g'(x)$ exists, we have

$$g'(x) = \lim_{a \to x} (x-a)^{-1}(g(x)-g(a))$$

$$= \lim_{a \to x} (x-a)^{-1} \sum_{n=1}^{\infty} \frac{-n^2(x-a)}{(1+n^2x)(1+n^2a)}$$

$$= \lim_{a \to x} \sum_{n=1}^{\infty} \frac{-n^2}{(1+n^2x)(1+n^2a)}$$

$$= \sum (-n^2)/(1+n^2x)^2 ,$$

which exists because it converges uniformly on $[a,\infty)$, as

$$\left|\frac{-n^2}{(1+n^2x)^2}\right| \le \left|\frac{n^2}{(n^2x)^2}\right| = \left|\frac{1}{n^2x^2}\right| \le \left|\frac{1}{a^2n^2}\right| := M_n$$

where $\sum M_n = \sum \frac{1}{a^2n^2} = \frac{1}{a^2}\sum\frac{1}{n^2} < \infty$.

So g is continuously differentiable on $(0,\infty)$. ■

(7a) <u>Claim</u>: $h_n \xrightarrow{u} 0$ on $[0, \infty)$

Note that $h_n'(x) = \dfrac{1-nx}{(1+x)^n}$ $\Rightarrow h_n' = 0$ iff $x = 1/n$ and

$$h_n''(x) = \frac{1+x+nx}{nx^2(1+x)^{n-1}} \quad \text{and} \quad h_n''(\tfrac{1}{n}) < 0,$$

so $x = \frac{1}{n}$ is a global maximum and thus

$$\forall x, \ |h_n(x)| \leq |h_n(\tfrac{1}{n})| = \left| \frac{1/n}{(1+\frac{1}{n})^n} \right| = \frac{1}{n(1+\frac{1}{n})^n} \leq \frac{1}{2n} \quad \text{for } n > 1$$

so $\sup_{x \in [0,\infty)} |h_n(x)| = |h_n(\tfrac{1}{n})| = O(\tfrac{1}{n}) \to 0$, thus $\|h_n\|_\infty \to 0$

and $h_n \to 0$ uniformly.

(7b) Let $h(x) = \sum_{n=1}^{\infty} h_n(x) = \sum_{n=1}^{\infty} x/(1+x)^{n+1}$

i) Demonstrably, $h(0) = 0$, and for a fixed x we have

$$h(x) = \sum_{n=1}^{\infty} x/(1+x)^{n+1} = (x/1+x) \sum_{n=1}^{\infty} (1/1+x)^n$$

$$= \frac{x}{1+x} \left(\frac{1}{1 - (1/1+x)} \right) \quad \text{since } x > 0 \Rightarrow (1/1+x) < 1$$

$$= 1. \quad \square$$

ii) It can <u>not</u> converge uniformly on $[0,\infty)$, otherwise h would be the uniform limit of continuous functions, but h is discontinuous.

7c) Let $a>0$ and $X=[a,\infty)$.

<u>Claim</u>: $\sum h_n \xrightarrow{u} h$ on X.

$(1+x)^{n+1} = \sum_{j=1}^{n} \binom{n}{j} x^j \geq \underbrace{1+nx+n^2x^2}$

$x>a>0$, so positive terms.

Since $x>a$, we have

$$|h_n(x)| = \left|\frac{x}{(1+x)^{n+1}}\right| \leq \left|\frac{x}{1+nx+n^2x^2}\right| \leq \left|\frac{a}{1+na+n^2a^2}\right| \leq \left|\frac{a}{n^2a^2}\right| = \left|\frac{1}{n^2a}\right|$$

So let $M_n = 1/an^2$, then $\sum M_n < \infty \Rightarrow \sum h_n \xrightarrow{u} h$ by the M test. ■

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(1) Suppose E is bounded, so $\text{diam}(E) \leq M$ for some fixed M. In particular, if $Q_i \subseteq E$ is an interval, then $|Q_i| \leq M$. Let $\varepsilon > 0$, and choose $\underbrace{\{Q_i\} \rightrightarrows E}_{\text{i.e. } E \subseteq \bigcup_i E_i}$ s.t.

$$\text{for each } i, \quad |Q_i| \leq \varepsilon/2M$$

Then let $L_i = Q_i^2$. We then have

$$|L_i| \leq |b^2 - a^2| = |b-a| \cdot |b+a| = |Q_i| \cdot |b+a|$$
$$\leq |Q_i| \cdot 2M$$
$$\leq (\varepsilon/2^{i+1}M) 2M$$
$$= \varepsilon/2^i,$$

so $\displaystyle\sum_{i=1}^\infty |L_i| \leq \sum_{i=1}^\infty \varepsilon/2^i = \varepsilon$, and $\{L_i\} \rightrightarrows E^2$, so

$$m_*(E^2) < \varepsilon \longrightarrow 0.$$

Claim: It suffices to consider the bounded case.

Pf If E is not bounded, consider $F_n = E \cap \underbrace{B(n,0)}_{\text{Ball of radius } n \text{ around } 0}$.

Then F_n is bounded (by n), and since $F_n \subseteq E \Rightarrow m_*(F_n) \leq m_*(E) = 0$ by subadditivity, $m_*(F_n^2) = 0$ by the bounded case.

But then $E^2 = \bigcup_{n=1}^{\infty} F_n^2 \Rightarrow m_*(E^2) = m\left(\bigcup_{n=1}^{\infty} F_n^2\right) \leq \sum_{n=1}^{\infty} m_*(F_n^2) = 0$

by countable subadditivity. ∎

---

② Note

1) $E_1 = E_1 \setminus E_2 \sqcup E_1 \cap E_2$

2) $E_2 = E_2 \setminus E_1 \sqcup E_1 \cap E_2$

3) $E_1 \triangle E_2 = E_2 \setminus E_1 \sqcup E_1 \setminus E_2$

4) $E_1 \cup E_2 = \underbrace{(E_1 \triangle E_2) \sqcup (E_1 \cap E_2)}$

All disjoint unions, so we can freely apply measures and use countable additivity.

so

$$
\begin{aligned}
m(E_1) + m(E_2) &= m(E_1 \setminus E_2) + m(E_1 \cap E_2) \\
&\quad + m(E_2 \setminus E_1) + m(E_1 \cap E_2) && \text{by (1), (2)} \\
&= \underbrace{m(E_1 \triangle E_2) + m(E_1 \cap E_2)} + m(E_1 \cap E_2) && \text{by (3)} \\
&= \underbrace{m(E_1 \cup E_2)} + m(E_1 \cap E_2). && \text{by (4)}
\end{aligned}
$$

∎

3a) Suppose $m(A)=m(B)<\infty$.

Since $A\subseteq E\subseteq B$, we have $E\setminus A\subseteq B\setminus A$. However,

$$B=A\sqcup(B\setminus A) \Rightarrow m(B)=m(A)+m(B\setminus A)$$

$$\Rightarrow m(B)-m(A)=m(B\setminus A)$$

(since $m(A)<\infty$)

$$\Rightarrow m(B\setminus A)=0$$

(since $m(B)=m(A)$)

So $m_*(E\setminus A)=0$ by subadditivity.

But then

$E=A\sqcup(E\setminus A)$, where A is measurable by assumption and $E\setminus A$ is an outer measure 0 set and thus measurable.

So E is measurable, and

$$m(E)=m(A)+m(E\setminus A)$$

$$=m(A)+0$$

$$\Rightarrow m(E)=m(A)=m(B)<\infty.$$

3b) Idea: $[0,1] \subseteq \mathcal{N} \subseteq [-1,2]$, so take

- $A = (-\infty, 0)$
- $E = A \cup (\mathcal{N}+1)$, where $\mathcal{N}$ is the non-measurable set, and $\mathcal{N}+1 = \{x+1 \mid x \in \mathcal{N}\}$ is non-measurable by the same argument used for $\mathcal{N}$.
- $B = \mathbb{R}$

Claim: E is not measurable.

Supposing it were, note that A^c is measurable, and countable intersections of measurable sets are measurable, so

$$E \cap A^c = (A \cup (\mathcal{N}+1)) \cap A^c = \mathcal{N}+1$$

must be measurable. ※

4) Let A, B be fixed, and define

$$E_t := \{x \in \mathbb{R}^n \mid \inf_{a \in A} |x-a| \leq t\} \cap B$$
$$= \{x \in \mathbb{R}^n \mid \mathrm{dist}(x, A) \leq t\} \cap B$$

and
$$f : \mathbb{R} \to \mathbb{R}$$
$$t \mapsto \mu(E_t)$$

Note that $E_0 = A$, so $f(0) = \mu(A)$, and since B is compact and thus bounded, there is some $t = T$ such that $B \subseteq E_T$.

So f maps $[0, T]$ to $[\mu(A), \mu(B) + M]$ for some M.

<u>Claim</u>: f is cts, and for all $t \in [0, T']$ for some T', $A \subseteq E_t \subseteq B$ and each E_t is compact.

Note that if this is true, we can first apply the intermediate value theorem to find a T' such that $f(T') = m(B)$, then restrict f to map $[0, T']$ to $[m(A), m(B)]$. We can apply it again to pull back any $c \in [m(A), m(B)]$ to a t satisfying $c = f(t) = \mu(E_t)$, in which case $A \subseteq E_t \subseteq B$ and $\mu(A) \leq c = \mu(E_t) \leq \mu(B)$ as desired.

• <u>f is cts</u>: We'll show that the 2-sided limit $\lim_{t_i \to t} f(t_i)$ exists and is equal to $f(t)$, using the fact that $a \leq b \Rightarrow E_a \subseteq E_b$.

If $t_i \nearrow t$, then $E_{t_1} \subseteq E_{t_2} \subseteq \cdots \subseteq E_t$, and $\bigcup_{i \in \mathbb{N}} E_{t_i} = E$, so by continuity of measure from below, we have $\lim_{i\to\infty} \mu(E_{t_i}) = \mu(E)$, so

$$\lim_{t_i\to t} f(t_i) = \lim_{i\to\infty} \mu(E_{t_i}) = \mu(E_t) = f(t).$$

Similarly, if $t_i \searrow t$, noting that $t_i \le T' \Rightarrow t_1 \le T' \Rightarrow \mu(E_{t_1}) \le \mu(B) < \infty$,

and $E_{t_1} \supseteq E_{t_2} \supseteq \cdots \supseteq E$, so

we can apply continuity of measure from above to obtain

$$\lim_{t_i\to t} f(t_i) = \lim_{i\to\infty} \mu(E_{t_i}) = \mu(E_t) = f(t).$$

So f is cts. ∎

- E_t is compact:

Since $E_t \subseteq B$ which is compact and thus bounded, it suffices to show that E_t is closed. But letting $N_t = \{x \in \mathbb{R}^n \mid \mathrm{dist}(x,A) < t\}$, we have $E_t = \overline{N_t \cap B}$, where N_t is open because $N_t = \bigcup_{a\in A} \underbrace{\{x\in\mathbb{R}^n \mid \mathrm{dist}(x,a) < r\}}_{\text{Open ball around } a}$, and $N_t \subseteq B \Rightarrow N_t \cap B$ is still open. But the closure of any open set is closed. ∎

- $t \in [0, T'] \Rightarrow A \subseteq E_t \subseteq B$:

$E_0 = A$ and $t \le s \Rightarrow E_t \subseteq E_s$, so $A \subseteq E_t$ for all t.

But $E_t = \overline{N_t \cap B} \subseteq \overline{B} = B$ since B is closed, so $E_t \subseteq B$ for all t as well. ∎

5a) Recalling that $\mathcal{N}$ is constructed by considering $\dfrac{\mathbb{R}\cap[0,1)}{\mathbb{Q}\cap[0,1)}$ and taking exactly one element from each equivalence class, we can note that if $E \subseteq \mathcal{N}$, then E contains a choice of <u>at most</u> one element from each equivalence class. We can then take a similar enumeration $\mathbb{Q}\cap[-1,1]=\{q_i\}_{i=1}^{\infty}$ and define $E_j := E + q_j$.

Then $E \subseteq \mathcal{N} \Rightarrow \bigsqcup_{j\in\mathbb{N}} E_j \subseteq \bigsqcup_{j\in\mathbb{N}} \mathcal{N}_j \subseteq [-1,2]$, and since E is measurable, we must have

$$\mu(E) = \mu\Big(\bigsqcup_{j\in\mathbb{N}} E_j\Big) = \sum_{j\in\mathbb{N}} \mu(E_j) = \sum_{j\in\mathbb{N}} \mu(E) \leq 3,$$

which can only hold if $m(E)=0$. $\square$

5b) Suppose $\mu(I\setminus\mathcal{N})<1$, so $m(I\setminus\mathcal{N})=1-2\varepsilon$ for some $\varepsilon>0$. Then choose an open $G \supseteq I\setminus\mathcal{N}$ such that $\mu(G)=\mu(I\setminus\mathcal{N})+\varepsilon = 1-\varepsilon$. Then $I\setminus G \subseteq \mathcal{N}$, and so by (1) we must have $\mu(I \setminus G) = 0$. But then

$$I = G \sqcup I \setminus G \Rightarrow \mu(I) = \mu(G) + \mu(I \setminus G)$$

$$\Rightarrow 1 = 1 - \varepsilon < 1,$$ a contradiction. □

5c) Let

$$\left.\begin{aligned} E_1 &= N \\ E_2 &= I \setminus N \end{aligned}\right\} \Rightarrow I = E_1 \sqcup E_2$$

but $m_*(E_1) = m_*(N) > 0$, otherwise N would be measurable so $m_*(E_1 \sqcup F_2) = 1$ but $m_*(E_1) + m_*(E_2) = 1 + \varepsilon$ for some $\varepsilon > 0$. ■

6a) Claim: E is a countable union of a countable intersection of measurable sets, and thus measurable.

Proof: Write $E = \{x \mid x \in E_j \text{ for infinitely many } j\}$, the claim is that

$$E = \bigcap_{j=1}^{\infty} \bigcup_{k=j}^{\infty} E_j.$$

• $E \subseteq \bigcap_{j=1}^{\infty} \bigcup_{k=j}^{\infty} E_j$: Suppose x is in infinitely many E_j. Then for any fixed k, there is some $M \geq k$ such that $x \in E_M \subseteq \bigcup_{j=k}^{\infty} E_j := S_k$. But this happens for every k, so $x \in \bigcap_{k=1}^{\infty} S_k$. $\square$

- $E \supseteq \bigcap_{j=1}^{\infty} \bigcup_{k=j}^{\infty} E_j$: Suppose $x \in \bigcup_{j=k}^{\infty} E_j$ for every k. Then if x were in only finitely many E_j, we could pick a maximal E_M such that $k \geq M \Rightarrow x \notin E_k$, and so $x \notin \bigcup_{j=M}^{\infty} E_j$ - a contradiction. $\square$

<u>Claim:</u> $m(E)=0$

We'll use the fact that $\sum_{n=1}^{\infty} a_n < \infty \Rightarrow \lim_{j\to\infty} \sum_{n=j}^{\infty} a_n = 0$, i.e. the tails of a convergent sum must become arbitrarily small.

Since $E = \bigcap_{k=1}^{\infty} \bigcup_{j=k}^{\infty} E_j$, $E \subseteq \bigcup_{j=k}^{\infty} E_j$ for all k. So $m(E) \leq \sum_{j=k}^{\infty} E_j \to 0$, forcing $m(E)=0$. $\blacksquare$

(6b) Fix x and let $E_{p,j} = \{ x \in \mathbb{R} \mid |x - p/j| \leq 1/j^3 \}$

and $E_j = \bigcup_{\substack{p \text{ coprime} \\ \text{to } j}} E_{p,j} \subseteq \bigcup_{p=1}^{j} E_{p,j}$, and since $E_{p,j} \subseteq B(1/j^3, p/j)$,

$m(E_{p,j}) \leq 2/j^3$ and thus $m(E_j) \leq q(2/j^3) = 2/j^2$.

But then $\sum_{j=1}^{\infty} m(E_j) \leq \sum_{j=1}^{\infty} 2/j^2 < \infty$. Moreover,

$$E = \bigcap_{j=1}^{\infty} \bigcup_{j=k}^{\infty} E_j = \{ x \in \mathbb{R} \mid \text{there are infinitely many } j\text{'s such that there exists a } p \text{ coprime to } j \text{ s.t. } |x - p/j| \leq 1/j^3 \},$$

which is precisely the set we want. So by (1), $m(E) = 0$. ∎

Analysis HW 3

Zack Garza

(1a) If $m_*(E)$, take $B = \mathbb{R}^n$, otherwise suppose $m_*(E) < \infty$ and let $\varepsilon > 0$. Choose $\{Q_i\} \rightrightarrows E$ then choose open $\{L_i\}$ s.t. $Q_i \subseteq L_i$ and $|L_i| < (m_*(E) + \varepsilon)/2^i$.

Then define $L(\varepsilon) = \bigcup_{i=1} L_i$; then $L(\varepsilon)$ is open (and thus Borel) and

$$m(L(\varepsilon)) = m_*(L(\varepsilon)) \leq \sum_{i=1}^{\infty} |L_i| < m_*(E) + \varepsilon.$$

So take the sequence $\varepsilon_k = 1/k \to 0$; then let $\boxed{L^n = \bigcap_{k=1}^{n} L_{1/k}}$. We have $L^{k+1} \subseteq L^k \ \forall k$, and $m(L^1) \leq m_*(E) + 1 < \infty$, so $L^n \searrow E$ and by upper continuity of measure,

$$m\left(\bigcap_{n=1}^{\infty} L^n\right) = m\left(\bigcap_{k=1}^{\infty} L_{1/k}\right) \overset{\text{continuity}}{=} \lim_{k\to\infty} m(L_{1/k}) = \lim_{k\to\infty} m_*(E) + 1/k = m_*(E),$$

so take $\boxed{B = \bigcap_{n=1}^{\infty} L^n.}$ ■

(1b) Let $\varepsilon > 0$; since $E \in \mathcal{L}(\mathbb{R}^n)$, there exists a closed set K_ε s.t. $m(E \setminus K_\varepsilon) < \varepsilon$. If $m(E) < \infty$, then $m(K_\varepsilon) = m(E) - \varepsilon$, so take the sequence $\varepsilon_n = 1/n$ and let $\boxed{K^n = \bigcup_{i=1}^{n} K_{1/i}}$, then $K^n \subseteq K^{n+1} \ \forall i$ and $K^n \nearrow E$, so by continuity of measure from below,

$$m\left(\bigcup_{n=1}^{\infty} K^n\right) = \lim_{n\to\infty} m(K^n) = \lim_{n\to\infty} m(E) - 1/n = m(E),$$

so take $\boxed{B = \bigcup_{n=1}^{\infty} K^n}$, which is a countable union of closed sets and thus Borel.

If $m(F) = \infty$, let $E_n = E \cap \overline{B(n,0)}$. Then $\exists B_n$ (by the bounded case) such that $B_n \subseteq E_n$ is closed and $m(B_n) = m(E_n)$. But $E_n \nearrow E$, so

$$m(E) = m\left(\bigcup_{n=1}^{\infty} E_n\right) = \lim_{n\to\infty} m(E_n) = \lim_{n\to\infty} m(B_n) = m\left(\bigcup_{n=1}^{\infty} B_n\right),$$

so take $B = \bigcup_{n=1}^{\infty} B_n$, which is borel since each B_n is. ■

(1c) Since $m(E) = m_*(E)$, choose $\{Q_j\} \rightrightarrows E$ closed cubes such that $\sum_{j=1}^{\infty} |Q_j| < m(E) + \varepsilon/2$.

Since $\sum_{i=1}^{\infty} |Q_i|$ converges, choose N such that $\sum_{i=N}^{\infty} |Q_i| < \varepsilon/2$, and let $\boxed{A = \bigcup_{i=1}^{N-1} Q_i.}$ Then,

$$E \Delta A = \underbrace{\left(E \setminus \bigcup_{i=1}^{N-1} Q_i\right)}_{} \coprod \underbrace{\left(\bigcup_{i=1}^{N-1} Q_i \setminus E\right)}_{}$$

$$\subseteq \bigcup_{i=N}^{\infty} Q_i \ \coprod \left(\bigcup_{i=1}^{\infty} Q_i \setminus E\right)$$

$$\Rightarrow m(E \Delta A) \leq m\left(\bigcup_{i=N}^{\infty} Q_i\right) + \left(m\left(\bigcup_{i=1}^{\infty} Q_i\right) - m(E)\right) \leq \varepsilon/2 + ((m(E) + \varepsilon/2) - m(E)) = \varepsilon. \ ■$$

(2a) Choose an open set $\mathcal{O} \supseteq E$ s.t. $m_*(\mathcal{O}) < (1/1-\varepsilon)\, m_*(E)$, so that $(1-\varepsilon)m_*(\mathcal{O}) < m_*(E)$.

Then write $\mathcal{O} = \bigsqcup_{i=1}^{\infty}{}^{\text{almost}}\, Q_i$ with each Q_i a closed cube, then towards a contradiction suppose that $m(E \cap Q_i) < (1-\varepsilon)m(Q_i)$ $\forall i$. Then, writing $E = \bigsqcup_{i=1}^{\infty}(E \cap Q_i)$, we have

$$m(E) = \sum_{i=1}^{\infty} m(E \cap Q_i) < \sum_{i=1}^{\infty}(1-\varepsilon)m(Q_i) = (1-\varepsilon)\, m\Big(\bigsqcup_{i=1}^{\infty} Q_i\Big) = (1-\varepsilon)m(\mathcal{O}) < m(E) \quad ※$$

So we must have $m(E \cap Q_j) \geq (1-\varepsilon)m(Q_j)$ for some j. ■

(2b) Let $\varepsilon > 0$ be arbitrary, and by (a) choose Q such that $m(E \cap Q) \geq (1-\varepsilon)m(Q)$.

Then let $E_0 = E \cap Q \subseteq E$, so $E_0 - E_0 \subseteq E - E$, and supposing towards a contradiction that $E_0 - E_0$ contains no ball around 0, choose $d \ll 1$ such that $d \notin E_0 - E_0$, and thus $E_0 \cap E_0 + d = \varnothing$. Also choose d small enough that $m(Q \cup Q+d) < m(Q) + \varepsilon$.

Then $E_0 \cup E_0 + d = E_0 \sqcup E_0 + d$, so $m(E_0 \cup E_0 + d) = 2m(E_0) \geq 2(1-\varepsilon)m(Q)$

Since $E_0 \cup E_0 + d \subseteq Q \cup Q + d$, we also have $m(E_0 \cup E + d) < m(Q) + \varepsilon$.

But then

$$2(1-\varepsilon)m(Q) \leq m(E_0 \cup E_0 + d) < m(Q) + \varepsilon$$

and taking $\varepsilon \to 0$ yields $2m(Q) < m(Q)$. ※

So $E_0 - E_0 \subseteq E - E$ must contain an open ball around 0. ■

---

(3) Fix x and let $L = \limsup_{y \to x} f(y) = \lim_{\delta \to 0} \sup_{y \in B_\delta(x)} f(y)$. Then consider $S_\alpha = \{x \in \mathbb{R}^n \mid f(x) \leq \alpha\}$;

we will show every $x \in S_\alpha$ has a ball $B_\delta(x) \subseteq S_\alpha$, making S_α open, and since α is arbitrary, this will show f is Borel measurable. Let $x \in S_\alpha$, so $f(x) < \alpha$. Then since f is upper-semicts, pick δ s.t. $y \in B_\delta(x) \Rightarrow f(y) \leq f(x)$. But then $y \in B_\delta(x) \Rightarrow f(y) \leq f(x) < \alpha \Rightarrow y \in S_\alpha$, so $B_\delta(x) \subseteq S_\alpha$ as desired. ■

---

(4) $S = \{x \in \mathbb{R}^n \mid \lim f_n(x) \text{ exists}\} \in \mathcal{M}$ iff $S^c \in \mathcal{M}$, which is what we'll show. Noting that if we let $F(x) = \limsup_{n\to\infty} f_n(x)$, $G(x) = \liminf_{n \to \infty} f_n(x)$, then

$$\begin{aligned} S^c &= \{x \mid F(x) > G(x)\} \\ &= \bigcup_{q \in \mathbb{Q}} \{x \mid F(x) > q > G(x)\} \\ &= \bigcup_{q \in \mathbb{Q}} \big(\{x \mid F(x) > q\} \cap \{x \mid G(x) < q\}\big) \end{aligned}$$

$= \bigcup_{q\in\mathbb{Q}} (M_q \cap N_q)$ where each M_q, N_q is measurable, thus making S^c a countable union of measurable sets & thus measurable. (E.g., M_q is measurable exactly because if $\{f_n\}$ are measurable, then $\limsup_{n\to\infty} f_n := F$ is measurable, as shown in class.) ∎

(5a) f is well-defined because each $x \in C$ has a unique ternary expansion which contains no 1's, and f is cts as we can write $g_n(x) = \underbrace{(a_n/2)\cdot(\tfrac{1}{2})^n}_{\text{cts}}$, so $f = \sum_{n=1}^{\infty} g_n$, where we have $|g_n(x)| \le 1/2^{n+1}$ which is summable, so f is uniformly cts by the M-test. Moreover,

$(0)_{10} = (0)_3 = (0.000\cdots)_3 \xmapsto{f} (0.000\cdots)_2 = (0)_{10}$, so $f(0)=0$, and

$(1)_{10} = (0.222\cdots)_3 \xmapsto{f} (0.111\cdots)_2 = (1)_{10}$, so $f(1)=1$. ∎

(5b) $f \twoheadrightarrow [0,1]$, so consider $f^{-1}(\mathcal{N})$ for $\mathcal{N}$ the non-measurable set. Since this is a subset of a measure zero set, it is measurable, and so $\underbrace{f^{-1}(\mathcal{N})}_{\text{measurable}} \underbrace{\xmapsto{f}}_{\text{cts}} \underbrace{\mathcal{N}}_{\text{not measurable}}$. ∎

(6a) Since f is cts, constant fns are cts, and f is a piecewise combination of cts fns that agree on intersections, F is cts. Constant fns are nondecreasing, so it only remains to show f is nondecreasing on C. Let $x = \sum a_n 3^{-n}$, $y = \sum b_n 3^{-n}$, and $x > y$. Then there is some minimal N such that $a_k = b_k \ \forall k < N$ and $a_N > b_N$. Then $\tfrac{1}{2}a_N > \tfrac{1}{2}b_N$, and $\tfrac{1}{2}a_k = \tfrac{1}{2}b_k \ \forall k<N$, which means that $f(x) > f(y)$ since

$$f(x) - f(y) = \sum_{n=1}^{\infty} (\tfrac{1}{2}a_n - \tfrac{1}{2}b_n)2^{-n} = \tfrac{1}{2}(a_N - b_N)2^{-N} + \tfrac{1}{2}\sum_{n=N+1}^{\infty}(a_n - b_n)2^{-n} \ge \tfrac{1}{2}(a_N - b_N)2^{-N} > 0.$$

(6b) Since $F(x)$ and $x \mapsto x$ are continuous and nondecreasing, and in fact $x \mapsto x$ is strictly increasing, G is continuous and strictly increasing & thus injective. To see that G is surjective, we just note that $G(0)=0$ and $G(1)=2$, so this follows from the IVT.

(6c1) Let I be one of the intervals in C^c, then $x, y \in I \Rightarrow F(x) = F(y)$ and so $G(b) - G(a) = b - a = m(I)$. Then $m(I) = m(G(I))$ since G is cts, and so $m(G(C^c)) = m(G(\bigsqcup_{n=1}^{\infty} I_n)) = m(\bigsqcup_{n=1}^{\infty} I_n) = 1$, so

$$m(G(C)) = m([0,2] \setminus G(C^c)) = 2 - 1 = 1.$$

(6c2) We have $\mathbb{R} = \bigsqcup_{q\in\mathbb{Q}} (\mathcal{N}+q)$, so $G(C) = \bigsqcup_{q\in\mathbb{Q}} (G(C) \cap \mathcal{N}+q)$, so $m(G(C)) \le \sum_{i=1}^{\infty} (G(C) \cap \mathcal{N}+q_i)$.

$$0 < 1 = m(G(C)) = \sum_{i=1}^{\infty} m\,(G(C) \cap \mathcal{N}+q_i).$$

Not every term can have $m_*(E_i)=0$, so some E_i has $m(E_i)>0$. But then E_i can not be measurable, since if we let $E_i = G(C)\cap \mathcal{N}+q_i$, then $x,y\in E_i \Rightarrow x-y\in \mathbb{R}\setminus\mathbb{Q}$, so $E_i - E_i$ can't contain any ball around zero and thus E can't be Lebesgue measurable by (2b). Since $E_i \subseteq G(C)$ is a nonmeasurable set, we're done.

(6c3) Let $\mathcal{N}' = E_i$, then $\mathcal{N}' = G(C)\cap \mathcal{N}+q_i$ for some i, so $G^{-1}(\mathcal{N}')\subseteq C$ and $m(C)=0$ implies $G^{-1}(\mathcal{N}')$ is measurable and $m(G^{-1}(\mathcal{N}'))=0$. But every cts function is Borel measurable, and since $G(G^{-1}(\mathcal{N}'))=\mathcal{N}'$ is not Borel, it can not pull back to a Borel set.

(6d) As shown above, E_i is not measurable and $G^{-1}(E_i)$ is null, so take $\varphi = \chi_{G^{-1}(E_i)}$. Then

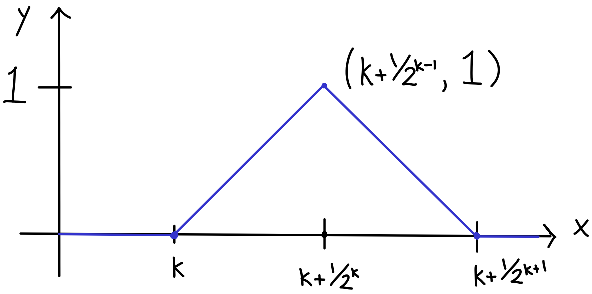
$$S_\alpha = \{x\in[0,1] \mid \varphi(x)>\alpha\} = \begin{cases} G^{-1}(E_i), & 0\le\alpha<1 \\ [0,1], & \alpha=0 \\ \emptyset, & \text{else} \end{cases}$$ both of which are measurable, so $\varphi\in\mathcal{M}$.

But for $\alpha=\frac{1}{2}$, $S_{\frac{1}{2}} = \{x\in[0,2] \mid (\varphi\circ G^{-1})(x)>\frac{1}{2}\} = \{x\in[0,2] \mid G^{-1}(x)\in G^{-1}(E_i)\} = E_i \notin \mathcal{M}$. ■

Analysis HW #4

Zack Garza

1a) Let f_k be the following function:



Note that this yields a triangle of area $\frac{1}{2}bh = \frac{1}{2}(k+\frac{1}{2^{k+1}} - k)\cdot 1 = 2^{-k}$, so we have $\int_{\mathbb{R}} f_k = \int_k^{k+\frac{1}{2^{k+1}}} f_k = 2^{-k}$. Moreover, $k \neq j \Rightarrow [k, k+\frac{1}{2^{k+1}}] \cap [j, j+\frac{1}{2^{j+1}}] = \emptyset$, so let $g_N = \sum_{k=0}^N f_k$ and $g = \lim_{N\to\infty} g_N = \sum_{k=0}^\infty f_k$. Then $g_N \nearrow g$, so we can apply the MCT to obtain

$$\int_{\mathbb{R}} g = \int_{\mathbb{R}} \lim_{N\to\infty} g_N \overset{\text{MCT}}{=} \lim_{N\to\infty} \int_{\mathbb{R}} g_N = \lim_{N\to\infty} \int_{\mathbb{R}} \sum_{k=0}^N f_k = \lim_{N\to\infty} \sum_{k=0}^N \int_{\mathbb{R}} f_k = \lim_{N\to\infty} \sum_{k=0}^N 2^{-k} = 1$$

Integral commutes with finite sums

However, $\limsup_{x\to\infty} g(x) = 1 > 0$, so $\lim_{x\to\infty} g(x) \neq 0$. ▨

1b) Towards a contradiction, suppose $f \in L^+$ is uniformly cts and $\limsup_{x\to\infty} f(x) = \varepsilon > 0$. Choose a sequence $\{x_n\} \nearrow \infty$ such that for all i,j we have $|x_i - x_j| > 1$. Then, for any $\delta < 1$ and any x_i, x_j, we have $B_\delta(x_i) \cap B_\delta(x_j) = \emptyset$. Now by uniform continuity of f, choose δ such that $\delta < 1$ and

$$y \in B_\delta(x) \Rightarrow |f(x) - f(y)| < \varepsilon \quad \forall x, y \in \mathbb{R}^N$$

Now let n be fixed, and consider some $x \in B_\delta(x_n)$. We have $|f(x) - f(x_n)| < \varepsilon$; note that $|f(x_n)| > 0$ for all n large enough; otherwise the limsup would be zero. It also must be the case that $|f(x)| > \varepsilon$; otherwise $|f(x)| < \varepsilon \Rightarrow \big||f(x_n)| - |f(x)|\big| > |0 - \varepsilon| = \varepsilon$, so

$$\varepsilon < \big||f(x_n)| - |f(x)|\big| \leq |f(x_n) - f(x)| < \varepsilon \quad \#$$

So $|f(x)| > \varepsilon$. But then

$$\int_{B_\delta(x_n)} |f| \geq \int_{B_\delta(x_n)} \varepsilon = \varepsilon \cdot m(B_\delta(x_n)) = \varepsilon \cdot 2\delta,$$

and so if we let

$$X = \bigsqcup_{n=1}^{\infty} B_{\delta}(x_n) \subsetneq \mathbb{R}^N,$$

we have

$$\int_{\mathbb{R}^N} |f| \geq \int_X |f| = \sum_{n=1}^{\infty} \int_{B_\delta(x_n)} |f| \leq \sum_{n=1}^{\infty} \varepsilon \cdot 2\delta \longrightarrow \infty,$$

contradicting $f \in L^1$. $\blacksquare$

2a) Let $X = \{ x \in \mathbb{R}^n \mid |f(x)| = \infty \}$, then $X \cap X^c = \emptyset$ and $\mathbb{R}^n = X \sqcup X^c$, so

$$\int_{\mathbb{R}^n} |f| = \int_X |f| + \int_{X^c} |f| = \infty \cdot m(X) + \int_{X^c} |f| < \infty$$

since $f \in L^1$; but if $m(X) > 0$ this yields a contradiction. So we must have $m(X) = 0$. $\blacksquare$

2b) We'll use the fact that $A \subseteq B$ and $\int_B |f| < \infty$, then $\int_B |f| - \int_A |f| = \int_{B \setminus A} |f|$. Noting that

$$\int_E |f| > \left(\int_{\mathbb{R}^n} |f| \right) - \varepsilon \iff \int_{\mathbb{R}^n} |f| - \int_E |f| < \varepsilon \iff \int_{E^c} |f| < \varepsilon,$$

we will produce an E s.t. E^c satisfies this condition. Write $\mathbb{R}^n = \lim_{k \to \infty} B(k, \vec{0})$, the n-ball of radius k centered at $\vec{0} \in \mathbb{R}^n$. Since the map $\left(A \mapsto \int_A |f|\right)$ is a measure, it satisfies continuity from below, and since $B(k, \vec{0}) \nearrow \mathbb{R}^n$, we have $\lim_{k \to \infty} \int_{B(k,\vec{0})} |f| = \int_{\mathbb{R}^n} |f|$. Since this limit exists, let $\varepsilon > 0$ and choose N such that

$$\int_{\mathbb{R}^n} |f| - \int_{B(N,\vec{0})} |f| < \varepsilon \implies \varepsilon > \int_{\mathbb{R}^n} |f| - \int_{B(N,\vec{0})} |f| = \int_{B(N,\vec{0})^c} |f|,$$

so $E := B(N, \vec{0})$ satisfies the desired property. $\blacksquare$

---

③ We want to show a iff b iff c, where

a) $\int f < \infty$

b) $\sum_{k \in \mathbb{Z}} 2^k m(E_k) < \infty, \qquad E_k = \{ x \mid f(x) > 2^k \}$

c) $\sum_{k \in \mathbb{Z}} 2^k m(F_k) < \infty, \qquad F_k = \{ x \mid 2^k < f(x) \leq 2^{k+1} \}$

Note that $F_i \cap F_j = \emptyset$ if $i \neq j$, and $F_k = E_k \setminus E_{k+1}$

(b) iff (c): We have

$$\begin{aligned}
\sum_{k\in\mathbb{Z}} 2^k m(F_k) &= \sum_{k\in\mathbb{Z}} 2^k [m(E_k) - m(E_{k+1})] \\
&= \sum_{k\in\mathbb{Z}} 2^k m(E_k) - \sum_{k\in\mathbb{Z}} 2^k m(E_{k+1}) \\
&= \sum_{k\in\mathbb{Z}} 2^k m(E_k) - \tfrac{1}{2}\sum_{k\in\mathbb{Z}} 2^{k+1} m(E_{k+1}) \\
&= \sum_{k\in\mathbb{Z}} 2^k m(E_k) - \tfrac{1}{2}\sum_{k\in\mathbb{Z}} 2^k m(E_k) \\
&= \sum_{k\in\mathbb{Z}} (1-\tfrac{1}{2}) 2^k m(E_k) \\
&= \tfrac{1}{2}\sum_{k\in\mathbb{Z}} 2^k m(E_k),
\end{aligned}$$

Might need to use absolute convergence of these sums for this to work.

and so either sum is finite iff the other is.

(a) $\Rightarrow$ (c) and (b) $\Rightarrow$ (a):

Write $X := \{x \mid f(x) > 0\} = \bigsqcup_{k\in\mathbb{Z}} F_k$, then $\int_X f = \sum_{k\in\mathbb{Z}} \int_{F_k} f$ and we have

$$\sum_{k\in\mathbb{Z}} 2^k m(F_k) \le \sum_{k\in\mathbb{Z}} \int_{F_k} f \le \sum_{k\in\mathbb{Z}} 2^{k+1} m(F_k) = \sum_{k\in\mathbb{Z}} 2^k m(E_k)$$

So

$$\int_X f < \infty \Rightarrow \sum_{k\in\mathbb{Z}} 2^k m(F_k) < \infty$$

and

$$\sum_{k\in\mathbb{Z}} 2^k m(E_k) < \infty \Rightarrow \int_X f < \infty.$$

$\blacksquare$

---

4) Let $A_k = \{x \in \mathbb{R}^n \mid 2^k < \|x\| \le 2^{k+1}\}$, so we have

$$A := \{x \in \mathbb{R}^n \mid \|x\| \le 1\} = \bigsqcup_{k=1}^{\infty} A_{-k}$$

$$B := \{x \in \mathbb{R}^n \mid \|x\| > 1\} = \bigsqcup_{k=0}^{\infty} A_k$$

$$\underbrace{\omega_n}_{\text{Volume of unit n-ball.}} 2^{nk} \le m(A_k) \le \omega_n 2^{n(k+1)}, \qquad \omega_n 2^{-nk} \le m(A_{(-k)}) \le \omega_n 2^{-n(k-1)}$$

Then noting that

$$x \in A_k \Rightarrow 2^k < \|x\| \le 2^{k+1} \Rightarrow 2^{-p(k+1)} \le \|x\|^{-p} < 2^{-kp},$$

$$x \in A_{(-k)} \Rightarrow 2^{-k} < \|x\| \le 2^{-(k-1)} \Rightarrow 2^{p(k-1)} \le \|x\|^{-p} < 2^{pk}$$

Raise to $-p$ power for $p>0$.

we define

(4a)

$$I_A = \int_A \|\vec{x}\|^{-p}, \quad I_B = \int_B \|\vec{x}\|^{-p}$$

and find

$$I_A \le \sum_{k=1}^{\infty} 2^{pk} m(A_{(-k)}) \le \sum_{k=1}^{\infty} 2^{pk}\, 2^{-n(k-1)} = \omega_n \sum_{k=1}^{\infty} (2^{-k})^{n-p} < \infty \quad \text{iff } p<n,$$

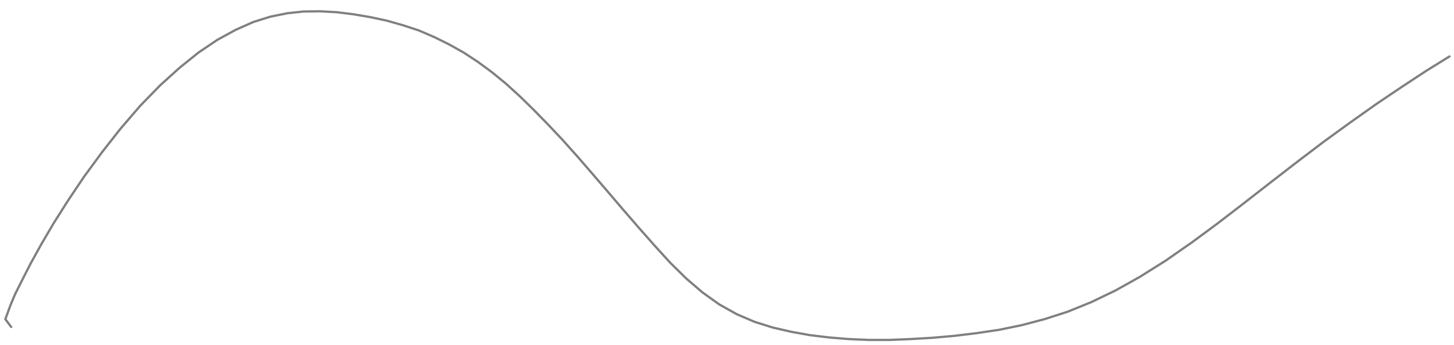
$$\text{and } \infty > I_A \ge \sum_{k=1}^{\infty} 2^{p(k-1)} m(A_{(-k)}) \ge \sum_{k=1}^{\infty} 2^{p(k-1)} \omega_n 2^{-nk} = \omega_n 2^{-p} \sum_{k=1}^{\infty} (2^{-k})^{n-p} \quad \text{iff } p<n$$

(4b)

Similarly

$$I_B \le \sum_{k=0}^{\infty} 2^{-kp} \omega_n 2^{n(k+1)} = \omega_n 2^{n} \sum_{k=0}^{\infty} (2^{-k})^{p-n} < \infty \quad \text{iff } p>n,$$

$$\text{and } \infty > I_B \ge \sum_{k=0}^{\infty} 2^{-p(k+1)} \omega_n 2^{nk} = \omega_n 2^{-p} \sum_{k=0}^{\infty} (2^{-k})^{p-n} \quad \text{iff } p>n. \quad \blacksquare$$



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⑤ To see that $\hat{f}$ is bounded, supposing that $f \in L^1(\mathbb{R}^n)$, we have

$$|\hat{f}(\xi)| \le \int |f(x)| \cdot \underbrace{|e^{2\pi i x\cdot\xi}|}_{\le 1} \le \int_{\mathbb{R}^n} |f| < \infty.$$

To see that it is cts, we will use the sequential defn. of continuity.

So let $\{\xi_n\} \to \xi$ be any sequence converging to ξ. Then

$$\lim_{n\to\infty} |\hat{f}(\xi_n) - \hat{f}(\xi)| = \lim_{n\to\infty} \left| \int f(x) [e^{2\pi i x\cdot\xi_n} - e^{2\pi i x\cdot\xi}] \right|$$

$$= \lim_{n\to\infty} \left| \int f(x) e^{2\pi i x\cdot\xi} [e^{2\pi i x\cdot(\xi_n - \xi)} - 1] \right|$$

$$\le \lim_{n\to\infty} \int |f(x) e^{2\pi i x\cdot\xi}| \cdot |e^{2\pi i x\cdot(\xi_n-\xi)} - 1|$$

$$\overset{DCT}{=} \int \lim_{n\to\infty} \left| f(x) e^{2\pi i x\cdot\xi} \right| \cdot \left| e^{2\pi i x\cdot(\xi_n - \xi)} - 1 \right|$$

Since

$$= \int \underbrace{\left| f(x) e^{2\pi i x\cdot\xi} \right|}_{\text{no } n \text{ involved}} \cdot \lim_{n\to\infty} \left| e^{2\pi i x\cdot(\xi_n - \xi)} - 1 \right|$$

$$= \int \left| f(x) e^{2\pi i x\cdot\xi} \right| \cdot 0$$

$$= 0$$

Where the DCT can be applied by letting

$$f_n = f(x) e^{2\pi i x\cdot\xi} \left(e^{2\pi i x\cdot(\xi_n - \xi)} - 1 \right)$$

$$\Rightarrow |f_n| = \left| f(x) e^{2\pi i x\cdot\xi} \right| \cdot \left| e^{2\pi i x\cdot(\xi_n - \xi)} - 1 \right|$$

$$\leq \left| f(x) e^{2\pi i x\cdot\xi} \right| \cdot \left(\underbrace{\left| e^{2\pi i x\cdot(\xi_n - \xi)} \right|}_{\leq 1} + |-1| \right)$$

$$\leq \left| f(x) e^{2\pi i x\cdot\xi} \right| \cdot 2$$

$$\leq 2|f| \in L^1.$$

But this says $\lim_{n\to\infty} |\hat{f}(\xi_n) - \hat{f}(\xi)| = 0$, so $\hat{f}$ is continuous. $\blacksquare$

6a.i) Let $g_n = |f_n| - |f_n - f|$; then $g_n \to |f|$ and

$$|g_n| = \big| |f_n| - |f_n - f| \big| \leq |f_n - (f_n - f)| = |f| \in L^1,$$

Reverse Δ-ineq

so $\lim_{n\to\infty} \int g_n = \int \lim_{n\to\infty} g_n = \int |f| = B$ by the DCT. We can then write

$$\lim_{n\to\infty} \int |f_n - f| = \lim_{n\to\infty} \int |f_n - f| - |f_n| + |f_n|$$

$$= \lim_{n\to\infty} \int |f_n| - \underbrace{(|f_n| - |f_n - f|)}_{:= g_n}$$

$$= \lim_{n\to\infty} \int |f_n| - g_n$$

$$= \lim_{n\to\infty} \int |f_n| - \lim_{n\to\infty} \int g_n = A - B \quad \blacksquare$$

6a.ii) Let $f_n = n\cdot\chi_{(0,\frac{1}{n}]}$, then $f_n \to 0 := f$ a.e., so $\int f = \int 0 = 0 \Rightarrow B = 0$, but $\int f_n = 1$ for all n, so $\lim_{n\to\infty} \int |f_n| = 1 = A \neq B$. ▨

6b) ($\Rightarrow$) $\lim_{k\to\infty} \int |f_k - f| = 0 = A - B \Rightarrow A = B \Rightarrow \lim \int |f_k| = \int |f|$.

($\Leftarrow$) $\lim \int |f_k| = \int |f| \Rightarrow A = B \Rightarrow A - B = 0 \Rightarrow \int |f_k - f| = A - B = 0$. ■

7a) Let $\{t_n\} \to t$ and define

$$g_n(x) = f(x)\left(\frac{\cos(t_n x) - \cos(tx)}{t_n - t}\right).$$

Then $\lim_{n\to\infty} g_n(x) = f(x)\, \partial/\partial t(\cos(tx)) = f(x)\, x \sin(tx)$, and applying the Mean Value Theorem, we have

$$\frac{\cos(t_n x) - \cos(tx)}{t_n - t} = x\sin(tx)\Big|_{x=\xi} = \xi \sin(t\xi) \text{ for some } \xi, \text{ so}$$

$$|g_n| = |f(x)\, x\sin(tx)| = |f(x)\, \xi \underbrace{\sin(t\xi)}_{\leq 1}| \leq \xi |f| \in L^1,$$

so $\lim_{n\to\infty} \int g_n \overset{DCT}{=} \int \lim_{n\to\infty} g_n = \int g = \int f(x)\, x\sin(tx)\, dx$, which is integrable because $\int |f(x)\, x \underbrace{\sin(tx)}_{\leq 1}| \leq \int |xf(x)| < \infty$ since $xf \in L^1$.

Thus $F'(t) = \int_{\mathbb{R}} f(x)\, x\sin(tx)\, dx$. ▨

7b)

$$\lim_{t\to 0} \int_0^1 \frac{e^{t\sqrt{x}} - 1}{t}\, dx = \lim_{t\to 0} \int_0^1 \frac{e^{t\sqrt{x}} - e^{0\sqrt{x}}}{t - 0}\, dx \overset{DCT}{=} \int_0^1 \lim_{t\to 0}\left(\frac{e^{t\sqrt{x}} - e^{0\sqrt{x}}}{t-0}\right) dx$$

$$:= \int_0^1 \frac{\partial}{\partial t} e^{t\sqrt{x}}\Big|_{t=0} dx = \int_0^1 \sqrt{x}\, e^{t\sqrt{x}}\Big|_{t=0} dx = \int_0^1 \sqrt{x}\, dx = (2/3)\, x^{3/2}\Big|_0^1 = 2/3.$$

The DCT here is justified by letting $\{t_n\} \to 0$ and setting $g_n(t) = \dfrac{e^{t\sqrt{x}} - e^{t_n\sqrt{x}}}{t - t_n}$

Then by the MVT, for each n we have $g_n(t) = \partial/\partial t\, e^{t\sqrt{x}}\Big|_{t=c}$ for some $c \in [0, t_n] \subseteq [0,1]$.

But $\partial/\partial t\, e^{t\sqrt{x}}\Big|_{t=c} = \sqrt{x}\, e^{t\sqrt{x}}\Big|_{t=c} = \sqrt{x}\, e^{c\sqrt{x}} \leq \sqrt{1}\, e^{c\cdot\sqrt{1}} = e^c \leq e^1$, so $|g_n| \leq e^1 \in L^1([0,1])$,

since $\int_0^1 e\, dx = e < \infty$, so $f(x) = e$ is a dominating function. ■

Problem Set 5

D. Zack Garza

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Contents

1 Problem 1

We first make the following claim:

$$\begin{aligned}
S \coloneqq \sum_{j=1}^\infty \sum_{k=1}^\infty a_{jk} &= \sup \left\{ \sum_{(j,k)\in B} a_{jk} \ni B \subset \mathbb{N}^2,\ |B| < \infty \right\} \\
T \coloneqq \sum_{k=1}^\infty \sum_{j=1}^\infty a_{kj} &= \sup \left\{ \sum_{(j,k)\in C} a_{kj} \ni C \subset \mathbb{N}^2,\ |B| < \infty \right\}.
\end{aligned}$$

It suffices to show the first equality holds, as the other case will follow similarly. Let $S = \sum_{j=1}^\infty \sum_{k=1}^\infty a_j k$ and $S' = \sup\left\{ \sum_{(j,k)\in B} a_{jk} \ni B \subset \mathbb{N}^2,\ |B| < \infty \right\}$.

Then consider any bounded set $B \subset \mathbb{N}^2$; so $B \subset \{1, \cdots, n_1\} \times \{1, \cdots, n_2\}$ for some $n_1, n_2 \in \mathbb{N}$. We then have

$$\sum_B a_{jk} \leq \sum_{j=1}^{n_1} \sum_{k=1}^{n_2} a_{jk} \leq \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} a_{jk}.$$

where the first equality holds $a + jk \geq 0$ for all j, k, so the sum can only increase if we add more terms. But this holds for every B and thus holds if we take the supremum over all of them, so $S' \leq S$.

To see that $S \leq S'$, we can just note that

$$\begin{aligned}
S &= \lim_{J \to \infty} \sum_{j=1}^{J} \left(\lim_{K \to \infty} \sum_{k=1}^{K} a_{jk} \right) \\
&= \lim_{J \to \infty} \lim_{K \to \infty} \sum_{j=1}^{J} \sum_{k=1}^{K} a_{jk} \\
&\leq \lim_{J \to \infty} \lim_{K \to \infty} S' \\
&= S',
\end{aligned}$$

where the limits commute with finite sums, and we the sum can be replaced with S' because the set $\{1, \cdots, K\} \times \{1, \cdots J\}$ is one of the finite sets over which the supremum is taken. Moreover, S' is a number that doesn't depend on J, K, yielding the final equality. $\square$

We will show that $S = T$ by showing that $S \leq T$ and $T \leq S$.

Let $B \subset \mathbb{N}^2$ be finite, so $B \subseteq [0, I] \times [0, J] \subset \mathbb{N}^2$.

Now letting $R > \max(I, J)$, we can define $C = [0, R]^2$, which satisfies $B \subseteq C \subset \mathbb{N}^2$ and $|C| < \infty$.

Moreover, since $a_{jk} \geq 0$ for all pairs (j, k), we have the following inequality:

$$\sum_{(j,k) \in B} a_{jk} < \sum_{(k,j) \in C} a_{jk} \leq \sum_{(k,j) \in C} a_{jk} \leq T,$$

since T is a supremum over *all* such sets C, and the terms of any finite sum can be rearranged.

But since this holds for every B, we this inequality also holds for the supremum of the smaller term by order-limit laws, and so

$$S := \sup_B \sum_{(k,j) \in B} a_{jk} \leq T.$$

(Use epsilon-delta argument)

An identical argument shows that $T \leq S$, yielding the desired equality. $\square$

2 Problem 2

We want to show the following equality:

$$\int_0^1 g(x)\ dx = \int_0^1 f(x)\ dx.$$

To that end, we can rewrite this using the integral definition of $g(x)$:

$$\int_0^1 \int_x^1 \frac{f(t)}{t}\ dt\ dx = \int_0^1 f(x)\ dx$$

Note that if we can switch the order of integration, we would have

$$\begin{aligned}
\int_0^1 \int_x^1 \frac{f(t)}{t}\ dt\ dx &=_? \int_0^1 \int_0^t \frac{f(t)}{t}\ dx\ dt \\
&= \int_0^1 \frac{f(t)}{t} \int_0^t\ dx\ dt \\
&= \int_0^1 \frac{f(t)}{t}(t-0)\ dt \\
&= \int_0^1 f(t)\ dt,
\end{aligned}$$

which is what we wanted to show, and so we are simply left with the task of showing that this is switch of integrals is justified.

To this end, define

$$\begin{aligned}
F : \mathbb{R}^2 &\to \mathbb{R} \\
(x,t) &\mapsto \frac{\chi_A(x,t)\hat{f}(x,t)}{t}.
\end{aligned}$$

where $A = \{(x,t) \subset \mathbb{R}^2 \ni 0 \leq x \leq t \leq 1\}$ and $\hat{f}(x,t) := f(t)$ is the cylinder on f.

This defines a measurable function on $\mathbb{R}^2$, since characteristic functions are measurable, the cylinder over a measurable function is measurable, and products/quotients of measurable functions are measurable.

In particular, $|F|$ is measurable and non-negative, and so we can apply Tonelli to $|F|$. This allows us to write

$$\begin{aligned}
\int_{\mathbb{R}^2} |F| &= \int_0^1 \int_0^t \left|\frac{f(t)}{t}\right|\ dx\ dt \\
&= \int_0^1 \int_0^t \frac{|f(t)|}{t}\ dx\ dt \quad \text{since } t > 0 \\
&= \int_0^1 \frac{|f(t)|}{t} \int_0^t\ dx\ dt \\
&= \int_0^1 |f(t)| < \infty,
\end{aligned}$$

where the switch is justified by Tonelli and the last inequality holds because f was assumed to be measurable.

Since this shows that $F \in L^1(\mathbb{R}^2)$, and we can thus apply Fubini to F to justify the initial switch. $\square$

3 Problem 3

Let $A = \{0 \leq x \leq y\} \subset \mathbb{R}^2$, and define

$$\begin{aligned} f(x,y) &= \frac{x^{1/3}}{(1+xy)^{3/2}} \\ F(x,y) &= \chi_A(x,y) f(x,y). \end{aligned}$$

Note that F Then, if all iterated integrals exist and a switch of integration order is justified, we would have

$$\begin{aligned} \int_{\mathbb{R}^2} F &=_? \int_0^\infty \int_y^\infty f(x,y)\ dx\ dy \\ &=_? \int_0^\infty \int_x^\infty \frac{x^{1/3}}{(1+xy)^{3/2}}\ dy\ dx \\ &= 2\int_{\mathbb{R}} \frac{1}{x^{2/3}\sqrt{1+x^2}}\ dx \\ &= 2\int_0^1 \frac{1}{x^{2/3}\sqrt{1+x^2}}\ dx + 2\int_1^\infty \frac{1}{x^{2/3}\sqrt{1+x^2}}\ dx \\ &\leq \int_0^1 x^{-2/3}\ dx + \int_0^\infty x^{-5/3} \\ &= 2(3) + 2\left(\frac{3}{2}\right) < \infty, \end{aligned}$$

where the first term in the split integral is bounded by using the fact that $\sqrt{1+x^2} \geq \sqrt{x^2} = x$, and the second term from $x > 1 \implies x > 0 \implies \sqrt{1+x^2} \geq \sqrt{1}$.

Since F is non-negative, we have $|F| = F$, and so the above computation would imply that $F \in L^1(\mathbb{R}^2)$. It thus remains to show that $\int F$ is equal to its iterated integrals, and that the switch of integration order is justified

Since F is non-negative, Tonelli can be applied directly if F is measurable in $\mathbb{R}^2$. But f is measurable on A, since it is continuous at almost every point in A, and χ_A is measurable, so F is a product of measurable functions and thus measurable.

4 Problem 4

4.1 Part (a)

For any $x \in \mathbb{R}^n$, let $A_x := A \bigcap (x - B)$.

We can then write $A_t \coloneqq A \bigcap (t-B)$ and $A_s \coloneqq A \bigcap (s-B)$, and thus

$$\begin{aligned}
g(t) - g(s) &= m(A_t) - m(A_s) \\
&= \int_{\mathbb{R}^n} \chi_{A_t}(x)\ dx - \int_{\mathbb{R}^n} \chi_{A_s}(x)\ dx \\
&= \int_{\mathbb{R}^n} \chi_{A_t}(x) - \chi_{A_s}(x)\ dx \\
&= \int_{\mathbb{R}^n} \chi_{A_t}(x) - \chi_{A_t}(t-s+x)\ dx \\
&\quad (\text{since } x \in s-B \iff s-x \in B \iff t-(s-x) \in t-B),
\end{aligned}$$

and thus by continuity in L^1, we have

$$|g(t) - g(s)| \leq \int_{\mathbb{R}^n} |\chi_{A_t}(x) - \chi_{A_t}(t-s+x)|\ dx \to 0 \quad \text{as} \quad t \to s$$

which means g is continuous.

To see that $\int g = m(A)m(B)$, if an interchange of integrals is justified, we can write

$$\begin{aligned}
\int_{\mathbb{R}^n} g(t)\ dt &= \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \chi_{A_t}(x)\ dx\ dt \\
&= \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \chi_A(x) \chi_{t-B}(x,t)\ dx\ dt \\
&= \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \chi_A(x) \chi_{t-B}(x,t)\ dx\ dt \\
&= \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \chi_A(x)\ \chi_B(t-x)\ dx\ dt \\
&\quad (\text{since } x \in t-B \iff t-x \in B) \\
&=_? \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \chi_A(x)\ \chi_B(t-x)\ \mathbf{dt}\ \mathbf{dx} \\
&= \int_{\mathbb{R}^n} \chi_A(x) \int_{\mathbb{R}^n} \chi_B(t-x)\ dt\ dx \\
&= \int_{\mathbb{R}^n} \chi_A(x)\ m(B)\ dt \\
&\quad (\text{by translation invariance of Lebesgue integral}) \\
&= m(B) \int_{\mathbb{R}^n} \chi_A\ dt \\
&= m(B)m(A).
\end{aligned}$$

To see that this is justified, we note that that the map $F(x,t) = \chi_A(x)\ \chi_B(x-t)$ is non-negative, and we claim is measurable in $\mathbb{R}^{2n}$.

- The first component is $\chi_A(x)$, which is measurable on $\mathbb{R}^n$, and thus the cylinder over it will be measurable on $\mathbb{R}^{2n}$.

- The second component involves $\chi_B(t-x)$, which is $\chi_B(x)$ composed with a reflection (which is still measurable) followed by a translation (which is again still measurable).
- Thus, as a product of two measurable functions, the integrand is measurable.

So Tonelli applies to $|F|$, and thus $\int |F| = m(A)m(B) < \infty$ since A, B were assumed to be bounded. But then F is integrable by Fubini, and the claimed equality holds.

4.2 Part (b)

Supposing that $m(A), m(B) > 0$, we have $\int g(t)\, dt > 0$, using the fact that $\int g = 0$ a.e. $\iff g = 0$ a.e., we can conclude that if $T = \{t \ni g(t) \neq 0\}$, then $m(T) > 0$. So there is some $t \in \mathbb{R}^n$ such that $g(t) \neq 0$, and since g is continuous, there is in fact some open ball B_t containing t such that $t' \in B_t \implies g(t') \neq 0$. So we have

- $\forall t' \in B_t,\ A \bigcap t' - B \neq \emptyset \iff$
- $\forall t' \in B_t,\ \exists x \in A \bigcap t' - B \iff$
- $\forall t' \in B_t,\ \exists x$ such that $x \in A$ and $x \in t' - B \iff$
- $\forall t' \in B_t,\ \exists x$ such that $x \in A$ and $x = t' - B$ for some $b \in B \iff$
- $\forall t' \in B_t,\ \exists x$ such that $x \in A$ and $t' = x + B$ for some $b \in B \iff$
- $\forall t' \in B_t,\ \exists t'$ such that $t' \in A + B$

And thus $B_t \subseteq A + B$.

5 Problem 5

If the iterated integrals exist and are equal (so an interchange of integration order is justified), we have

$$\begin{aligned}
\int_0^1 F(x)g(x) &:= \int_0^1 \left(\int_0^x f(y)\ dy \right) g(x)\ dx \\
&= \int_0^1 \int_0^x f(y)g(x)\ dy\ dx \\
&=_? \int_0^1 \int_y^1 f(y)g(x)\ \mathbf{dx\ dy} \\
&= \int_0^1 f(y) \left(\int_y^1 g(x)\ dx \right)\ dy \\
&= \int_0^1 f(y)(G(1) - G(y))\ dy \\
&= G(1) \int_0^1 f(y)\ dy - \int_0^1 f(y)G(y)\ dy \\
&= G(1)(F(1) - F(0)) - \int_0^1 f(y)G(y)\ dy \\
&= G(1)F(1) - \int_0^1 f(y)G(y)\ dy \qquad \text{since } F(0) = 0,
\end{aligned}$$

which is what we want to show.

To see that this is justified, let $I = [0,1]$ and note that the integrand can be written as $H(x,y) = \hat{f}(x,y)\hat{g}(x,y)$ where $\hat{f}(x,y) = \chi_I f(y)$ and $\hat{g}(x,y) = \chi_I g(x)$ are cylinders over f and g respectively. Since f, g are in $L^1(I)$, their cylinders are measurable over $\mathbb{R} \times I$, and thus $\hat{f}, \hat{g}$ are measurable on $\mathbb{R}^2$ as products of measurable functions. Then H is a measurable function as a product of measurable functions as well.

But then $|H|$ is non-negative and measurable, so by Tonelli all iterated integrals will be equal. We want to show that $H \in L^1(\mathbb{R}^2)$ in order to apply Fubini, so we will show that $\int |H| < \infty$.

To that end, noting that $f, g \in L^1$, we have $\int_0^1 f \coloneqq C_f < \infty$ and $\int_0^1 g \coloneqq C_g < \infty$. Then,

$$\begin{aligned}
\int_{\mathbb{R}^2} |H| &= \int_0^1 \int_0^1 |f(x)g(y)| \; dx \; dy \\
&= \int_0^1 \int_0^1 |f(x)| \; |g(y)| \; dx \; dy \\
&= \int_0^1 |g(y)| \left(\int_0^1 |f(x)| \; dx \right) \; dy \\
&= \int_0^1 |g(y)| C_f \; dy \\
&= C_f \int_0^1 |g(y)| \; dy \\
&= C_f C_g < \infty,
\end{aligned}$$

and thus by Fubini, the original interchange of integrals was justified.

6 Problem 6

6.1 Part (a)

We have

$$
\begin{aligned}
\int_{\mathbb{R}} |A_h(f)(x)|\ dx &= \int_{\mathbb{R}} \left| \frac{1}{2h} \int_{x-h}^{x+h} f(y)\ dy \right| dx \\
&= \frac{1}{2h} \int_{\mathbb{R}} \left| \int_{x-h}^{x+h} f(y)\ dy \right| dx \\
&\leq \frac{1}{2h} \int_{\mathbb{R}} \left(\int_{x-h}^{x+h} |f(y)|\ dy \right) dx \\
&= \frac{1}{2h} \int_{\mathbb{R}} \int_{x-h}^{x+h} |f(y)|\ dy\ dx \\
&=_? \frac{1}{2h} \int_{\mathbb{R}} \int_{y-h}^{y+h} |f(y)|\ \mathbf{dx\ dy} \\
&= \frac{1}{2h} \int_{\mathbb{R}} |f(y)| \int_{y-h}^{y+h}\ dx\ dy \\
&= \frac{1}{2h} \int_{\mathbb{R}} |f(y)| \left((y+h) - (y-h) \right)\ dy \\
&= \frac{1}{2h} \int_{\mathbb{R}} 2h|f(y)|\ dy \\
&= \int_{\mathbb{R}} |f(y)|\ dy < \infty
\end{aligned}
$$

.

since f was assumed to be in $L^1(\mathbb{R})$, where the changed bounds of integration are determined by considering the following diagram:

To justify the change in the order of integration, consider the function $H(x,y) = \frac{1}{2h}\chi_A(x,y)f(y)$ where $A = \left\{(x,y) \in \mathbb{R}^2 \ni -\infty < x-h \leq x, y \leq x+h\right\}$. Since f is measurable, the constant function $(x,y) \mapsto \frac{1}{2h}$ is measurable, and characteristic functions are measurable, H is a product of measurable functions and thus measurable.

Thus it makes sense to write $\int |H|$ as an iterated integral by Tonelli, and since $\int_{\mathbb{R}^2} |H| = \int_{\mathbb{R}} |A_h(f)| < \infty$ by the above calculation, we have $H \in L^1(\mathbb{R}^2)$, and Fubini applies.

6.2 Part (b)

Let $\varepsilon > 0$; we then have

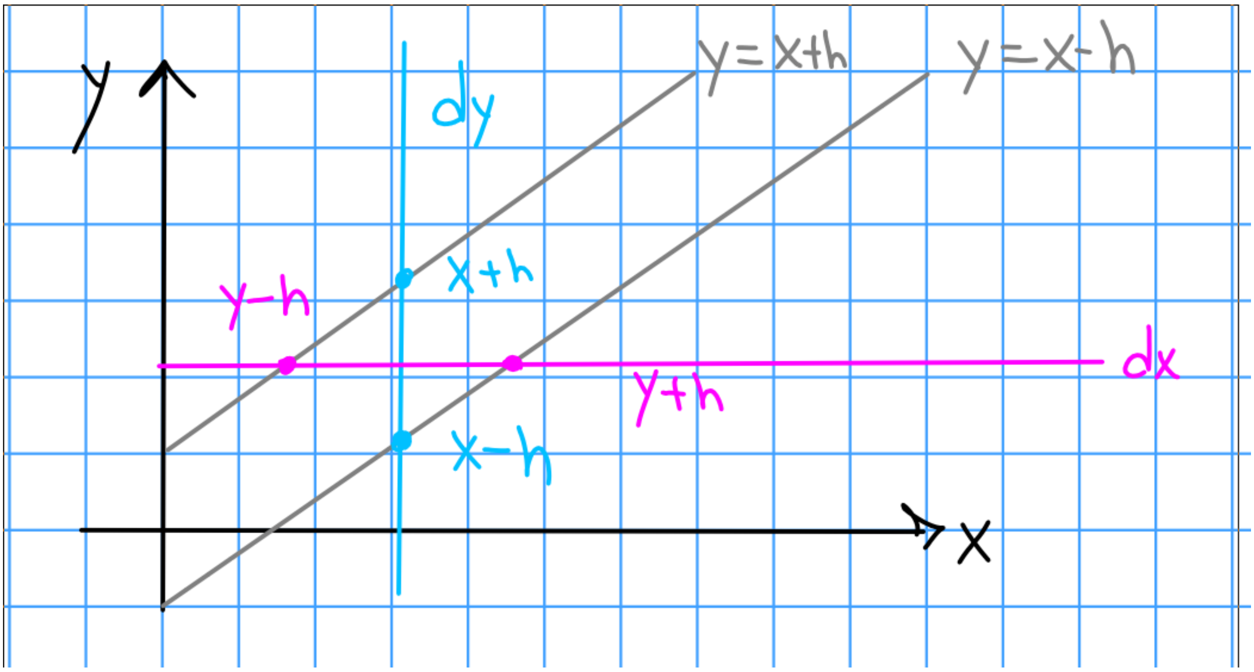

Figure 1: Changing the bounds of integration

$$
\begin{aligned}
\int_{\mathbb{R}} |A_h(f)(x) - f(x)| \, dx &= \int_{\mathbb{R}} \left| \left(\frac{1}{2h} \int_{B(h,x)} f(y) \, dy \right) - f(x) \right| \, dx \\
&= \int_{\mathbb{R}} \left| \left(\frac{1}{2h} \int_{B(h,x)} f(y) \, dy \right) - \frac{1}{2h} \int_{B(h,x)} f(x) \, dy \right| \, dx \\
&\quad \text{since } \frac{1}{2h} \int_{x-h}^{x+h} f(x) \, dy = \frac{1}{2h} f(x)((x+h) - (x-h)) = \frac{1}{2h} f(x) 2h = f(x) \\
&= \int_{\mathbb{R}} \left| \frac{1}{2h} \int_{B(h,x)} f(y) - f(x) \, dy \right| \, dx \\
&\leq \int_{\mathbb{R}} \frac{1}{2h} \int_{x-h}^{x+h} |f(y) - f(x)| \, dy \, dx \\
&\leq \int_{\mathbb{R}} \frac{1}{2h} \int_{-h}^{h} |f(y-x) - f(x)| \, dy \, dx
\end{aligned}
$$

.

but since $h \to 0$ will force $y \to x$ in the integral, for a fixed x we can let $\tau_x(y) = f(y-x)$ and we have $\|\tau_x - f\|_1 \to 0$ by continuity in L^1. Thus $\int_{-h}^{h} |f(y-x) - f(x)| \to 0$, forcing $\|A_h(f) - f\|_1 \to 0$ as $h \to 0$. $\square$

Assignment 6: The Fourier Transform

D. Zack Garza

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Contents

1 Problem 1

Assuming the hint, we have

$$\lim_{|\xi|\to\infty} \hat{f}(\xi) = \lim_{|\xi'|\to 0} \frac{1}{2} \int_{\mathbb{R}^n} [f(x)) - f(x-\xi')]\, e^{-2\pi i x\cdot\xi}\, dx$$

.

The fact that the limit as $\xi \to \infty$ is equivalent to the limit $\xi' \to 0$ is a direct consequence of computing

$$\lim_{|\xi|\to\infty} \frac{\xi}{2|\xi|^2} = \lim_{|\xi|\to\infty} \frac{1}{2|\xi|}\frac{\xi}{|\xi|} = \mathbf{0},$$

since $\frac{\xi}{|\xi|}$ is a unit vector, and the term $\frac{1}{2|\xi|}$ is a scalar that goes to zero.

But as an immediate consequence, this yields

$$\begin{aligned}
\left|\hat{f}(\xi)\right| &= \frac{1}{2}\left|\int_{\mathbb{R}^n} [f(x) - f(x-\xi')]\, e^{-2\pi i x\cdot\xi}\, dx\right| \\
&\leq \int_{\mathbb{R}^n} |f(x) - f(x-\xi')|\, \left|e^{-2\pi i x\cdot\xi}\right|\, dx \\
&\leq \int_{\mathbb{R}^n} |f(x) - f(x-\xi')|\, dx \\
&\to 0,
\end{aligned}$$

which follows from continuity in L^1 since $f(x-\xi') \to f(x)$ as $\xi' \to 0$.

It thus only remains to show that the hint holds.

Note: Sorry, I couldn't figure out how to prove the hint!!

2 Problem 2

2.1 Part (a)

Assuming an interchange of integrals is justified, we have

$$\begin{aligned}
\widehat{f * g}(\xi) &:= \int \int f(x-y) g(y)\ e^{-2\pi i x \cdot \xi}\ dy\ dx \\
&=_? \int \int f(x-y) g(y)\ e^{-2\pi i x \cdot \xi}\ dx\ dy \\
&= \int \int f(t) e^{-2\pi i (x-y) \cdot \xi}\ g(y)\ e^{-2\pi i y \cdot \xi}\ dx\ dy \\
&\quad (t = x-y,\ dt =\ dx) \\
&= \int \int f(t) e^{-2\pi i t \cdot \xi} g(y) e^{-2\pi i y \cdot \xi}\ dt\ dy \\
&= \int f(t) e^{-2\pi i t \cdot \xi} \left(\int\ g(y)\ e^{-2\pi i y \cdot \xi}\ dy \right)\ dt \\
&= \int f(t) e^{-2\pi i t \cdot \xi}\ \hat{g}(\xi)\ dt \\
&= \hat{g}(\xi) \int f(t) e^{-2\pi i t \cdot \xi}\ dt \\
&= \hat{g}(\xi) \hat{f}(\xi).
\end{aligned}$$

To see that this swap is justified, we'll apply Fubini-Tonelli. Note that if $f, g \in L^1(\mathbb{R}^n)$, then the map $(x, y) \mapsto f(x-y)$ is measurable on $\mathbb{R}^n \times \mathbb{R}^n$. Since g is measurable as well, taking the cylinder on g is also measurable on $\mathbb{R}^n \times \mathbb{R}^n$. The exponential is continuous, and thus measurable on $\mathbb{R}^n$. Thus the integrand $F(x, y)$ is a product of measurable functions and thus measurable. In particular, $|F| = |fg|$ is measurable, and ths computation shows that one iterated integral is finite. From a previous homework question, we know that $f \in L^1 \implies \hat{f}$ is bounded, and thus $\hat{f}\hat{g}$ is bounded. Since $|F|$ is measurable and one iterated integrable was finite, Fubini-Tonelli applies.

2.2 Part (b)

We'll use the following lemma: if $\hat{f} = \hat{g}$, then $f = g$ almost everywhere.

2.2.1 (i)

By part 1, we have

$$\widehat{f * g} = \hat{f}\hat{g} = \hat{g}\hat{f} = \widehat{g * f},$$

and so by the lemma, $f * g = g * f$.

Similarly, we have

$$\widehat{(f * g) * h} = \widehat{f * g}\ \hat{h} = \hat{f}\ \hat{g}\ \hat{h} = \hat{f}\ \widehat{g * h} = f * (g * h).$$

2.2.2 (ii)

Suppose that there exists some $I \in L^1$ such that $f * I = f$. Then $\widehat{f * I} = \hat{f}$ by the lemma, so $\hat{f}\ \hat{I} = \hat{f}$ by the above result.

But this says that $\hat{f}(\xi)\hat{I}(\xi) = \hat{f}(\xi)$ almost everywhere, and thus $\hat{I}(\xi) = 1$ almost everywhere. Then

$$\lim_{|\xi|\to\infty} \hat{I}(\xi) \neq 0,$$

which by Problem 1 shows that I can not be in L^1, a contradiction.

3 Problem 3

3.1 (a)

3.1.1 (i)

Let $g(x) = f(x-y)$. We then have

$$\begin{aligned}
\hat{g}(\xi) &:= \int g(x)e^{-2\pi i x\cdot\xi}\ dx \\
&= \int f(x-y)e^{-2\pi i x\cdot\xi}\ dx \\
&= \int f(x-y)e^{-2\pi i(x-y)\cdot\xi}e^{-2\pi i y\cdot\xi}\ dx \\
&= e^{-2\pi i y\cdot\xi}\int f(x-y)e^{-2\pi i(x-y)\cdot\xi}\ dx \\
&\quad (t = x-y, dt = dx) \\
&= e^{-2\pi i y\cdot\xi}\int f(t)e^{-2\pi i t\cdot\xi}\ dt \\
&= e^{-2\pi i y\cdot\xi}\hat{f}(\xi).
\end{aligned}$$

3.1.2 (ii)

Let $h(x) = e^{2\pi i x\cdot y}f(x)$. We then have

$$\begin{aligned}
\hat{h}(\xi) &:= \int e^{2\pi i x\cdot y}f(x)e^{-2\pi i x\cdot\xi}\ dx \\
&= \int e^{2\pi i x\cdot y - 2\pi i x\cdot\xi)f(x}\ dx \\
&= \int f(\xi-y)e^{-2\pi i x\cdot(\xi-y)}\ dx \\
&= \hat{f}(\xi-y).
\end{aligned}$$

3.2 (b)

We'll use the fact that if $\langle\cdot,\ \cdot\rangle$ is an inner product on a vector space V and A is an invertible linear transformation, then for all $\mathbf{x}, \mathbf{y} \in V$ we have

$$\langle A\mathbf{x},\ \mathbf{y}\rangle = \left\langle \mathbf{x},\ A^T\mathbf{y}\right\rangle$$

where A^{-T} denotes the transpose of the inverse of A (or $(A^{-1})^*$ if V is complex).

We then have

$$\begin{aligned}
\frac{1}{|\det T|}\hat{f}(T^{-T}\xi) &= \frac{1}{|\det T|}\int f(x)e^{-2\pi i x\cdot T^{-T}\xi}\ dx \\
&\qquad x \mapsto Tx,\ dx \mapsto |\det T|\ dx \\
&= \frac{1}{|\det T|}\int f(Tx)e^{-2\pi i Tx\cdot T^{-T}\xi}|\det T|\ dx \\
&= \int f(Tx)e^{-2\pi i x\cdot\xi}\ dx \\
&\qquad \text{since } Tx\cdot T^{-T}\xi = T^{-1}Tx\cdot\xi = x\cdot\xi \\
&= \widehat{(f\circ T)}(\xi).
\end{aligned}$$

4 Problem 4

4.1 (a)

4.1.1 (i)

Let $g(x) = xf(x)$. Then if an interchange of the derivative and the integral is justified, we have

$$\begin{aligned}
\frac{\partial}{\partial\xi}\hat{f}(\xi) &:= \frac{\partial}{\partial\xi}\int f(x)e^{-2\pi i x\cdot\xi}\ dx \\
&=_? \int f(x)\frac{\partial}{\partial\xi}e^{-2\pi i x\cdot\xi}\ dx \\
&= \int f(x)2\pi i x e^{-2\pi i x\cdot\xi}\ dx \\
&= 2\pi i\int xf(x)e^{-2\pi i x\cdot\xi}\ dx \\
&:= 2\pi i\hat{g}(\xi).
\end{aligned}$$

To see that the interchange is justified, we just note that we can apply the dominated convergence theorem, since $\int \left|f(x)e^{-2\pi i x\cdot\xi}\right| \leq \int |f| < \infty$, where we assumed $f \in L^1$.

4.1.2 (ii)

We have

$$\begin{aligned}
\hat{h}(\xi) &:= \int \frac{\partial f}{\partial x}(x) e^{-2\pi i x\cdot\xi}\ dx \\
&= f(x)e^{-2\pi i x\cdot\xi}\Big|_{x=-\infty}^{x=\infty} - \int f(x)(2\pi i\xi)e^{-2\pi i x\cdot\xi}\ dx \\
&\qquad \text{(integrating by parts)} \\
&= -\int f(x)(-2\pi i\xi)e^{-2\pi i x\cdot\xi}\ dx \\
&\qquad \text{(since } f(\infty) = f(-\infty) = 0\text{)} \\
&= 2\pi i\xi \int f(x)e^{-2\pi i x\cdot\xi}\ dx \\
&:= 2\pi i\xi\hat{f}(\xi).
\end{aligned}$$

4.2 (b)

Let $G(x) = e^{-\pi x^2}$ and ∂_ξ be the operator that differentiates with respect to ξ.

Then

$$\partial_\xi\left(\frac{\hat{G}(\xi)}{G(\xi)}\right) = \frac{G(\xi)\partial_\xi\hat{G}(\xi) - \hat{G}(\xi)\partial_\xi G(\xi)}{G(\xi)^2},$$

and the claim is that this is zero. This happens precisely when the numerator is zero, so we'd like to show that

$$G(\xi)\partial_\xi\hat{G}(\xi) - \hat{G}(\xi)\partial_\xi G(\xi) = 0.$$

A direct computation shows that

$$\partial_\xi G(\xi) = -2\pi\xi G(\xi), \tag{1}$$

and we claim that $\partial_\xi\hat{G}(\xi) = -2\pi\xi\hat{G}(\xi)$ as well, which follows from the following computation:

$$\begin{aligned}
\partial_\xi \hat{G}(\xi) &:= \partial_\xi \int G(x) e^{-2\pi i x\cdot\xi}\ dx \\
&= \int G(x) \partial_\xi e^{-2\pi i x\cdot\xi}\ dx \\
&= \int G(x)(-2\pi i x) e^{-2\pi i x\cdot\xi}\ dx \\
&= \int G(x)(-2\pi i x) e^{-2\pi i x\cdot\xi}\ dx \\
&= i \int 2\pi x G(x) e^{-2\pi i x\cdot\xi}\ dx \\
&= i \int \partial_x G(x) e^{-2\pi i x\cdot\xi}\ dx \qquad \text{by (1)} \\
&:= i\ \widehat{\partial_x G(x)}(\xi) \\
&= i\ (2\pi i \xi \hat{G}(\xi)) \qquad \text{by part (i)} \\
&= -2\pi\xi \hat{G}(\xi).
\end{aligned}$$

We can thus write

$$G(\xi)\partial_\xi \hat{G}(\xi) - \hat{G}(\xi)\partial_\xi G(\xi) = G(\xi)(-2\pi\xi\hat{G}(\xi)) - \hat{G}(\xi)(-2\pi\xi G(\xi)),$$

which is patently zero.

It follows that $\frac{\hat{G}(\xi)}{G(\xi)} = c_0$ for some constant c_0, from which it follows that $\hat{G}(\xi) = c_0 G(\xi)$.

Using the fact that $G(0) = 1$ by direct evaluation and $\hat{G}(0) = \int G(x)\ dx = 1$, we can conclude that $c_0 = 1$ and thus $\hat{G}(\xi) = G(\xi)$.

5 Problem 5

5.1 (a)

By a direct computation. we have

$$
\begin{aligned}
\hat{D}(\xi) &:= \int_{-\frac{1}{2}}^{\frac{1}{2}} 1 e^{-2\pi i x\xi}\ dx \\
&= \int_{-\frac{1}{2}}^{\frac{1}{2}} \cos(-2\pi x\xi) + i\sin(-2\pi x\xi)\ dx \\
&= \int_{-\frac{1}{2}}^{\frac{1}{2}} \cos(-2\pi x\xi)\ dx \\
&\qquad \text{(since sin is odd and the domain is symmetric about 0)} \\
&= 2\int_{0}^{\frac{1}{2}} \cos(-2\pi x\xi)\ dx \\
&\qquad \text{(since cos is even and the domain is symmetric about 0)} \\
&= 2\left(\frac{1}{2\pi\xi}\sin(-2\pi x\xi)\Big|_{x=0}^{x=\frac{1}{2}}\right) \\
&= \frac{\sin(\pi\xi)}{\pi\xi}.
\end{aligned}
$$

5.2 (b)

5.2.1 (i)

Since $F(x) = D(x) * D(x)$, we have $\hat{F}(\xi) = (\hat{D}(\xi))^2$ by question 2a, and so $\hat{F}(\xi) = \left(\frac{\sin(\pi\xi)}{\pi\xi}\right)^2$.

5.2.2 (ii)

Letting $\mathcal{F}$ denote the Fourier transform operator, we have $\mathcal{F}^2(h)(\xi) = h(-\xi)$ for any $h \in L^1$. In particular, if f is an even function, then $f(\xi) = -f(\xi)$ and $\mathcal{F}^2(f) = f$.

In this case, letting F be the box function, F can be seen to be even from its definition. Since $f := \mathcal{F}(F)$ by part (i), we have

$$\hat{f} := \mathcal{F}(f) = \mathcal{F}(\mathcal{F}(F)) = \mathcal{F}^2(F) = F,$$

which says that $\hat{f}(x) = F(x)$, the original box function.

5.3 (c)

By a direct computation of the integral in question, we have

$$
\begin{aligned}
I(x) &\coloneqq \int e^{-2\pi|\xi|}e^{2\pi ix\xi}\ d\xi \\
&= \int_{-\infty}^{0} e^{-2\pi(-\xi)}e^{-2\pi ix\xi}\ d\xi + \int_{0}^{\infty} e^{2\pi\xi}e^{2\pi ix\xi}\ d\xi \\
&= \int_{0}^{\infty} e^{-2\pi\xi}e^{-2\pi ix\xi}\ d\xi + \int_{0}^{\infty} e^{2\pi\xi}e^{2\pi ix\xi}\ d\xi \\
&\qquad \text{by the change of variables } \xi \mapsto -\xi,\ d\xi \mapsto -d\xi \text{ and swapping integration bounds} \\
&= \int_{0}^{\infty} e^{-2\pi\xi}e^{-2\pi ix\xi} + e^{2\pi\xi}e^{2\pi ix\xi}\ d\xi \\
&= \frac{1}{2\pi}\int_{0}^{\infty} e^{-u}e^{-ixu} + e^{-u}e^{ixu}\ du \\
&= \frac{1}{2\pi}\int_{0}^{\infty} e^{-u(1+ix)} + e^{-u(1-ix)}\ du \\
&= \frac{1}{2\pi}\left(\frac{-e^{-u(1+ix)}}{1+ix}\Big|_{u=0}^{u=\infty} + \frac{-e^{-u(1-ix)}}{1+ix}\Big|_{u=0}^{u=\infty}\right) \\
&= \frac{1}{2\pi}\left(\frac{1}{1+ix} + \frac{1}{1-ix}\right) \\
&= \frac{1}{2\pi}\frac{2}{1+x^2} \\
&= \frac{1}{\pi}\frac{1}{1+x^2},
\end{aligned}
$$

so $P(x) = I(x)$.

Then, by the Fourier inversion formula, we have

$$
\begin{aligned}
I(x) = P(x) &= \int \hat{P}(\xi)e^{-2\pi ix\xi}\ dx \\
\implies \int e^{-2\pi|\xi|}e^{2\pi ix\xi} &= \int \hat{P}(\xi)e^{-2\pi ix\xi}\ dx \\
\implies \int e^{-2\pi|\xi|}e^{2\pi ix\xi} - \hat{P}(\xi)e^{-2\pi ix\xi}\ dx &= 0 \\
\implies \int \left(e^{-2\pi|\xi|} - \hat{P}(\xi)\right)e^{-2\pi ix\xi}\ dx &= 0 \\
\implies \left(e^{-2\pi|\xi|} - \hat{P}(\xi)\right)e^{-2\pi ix\xi} &=_{a.e.} 0 \\
\implies e^{-2\pi|\xi|} &=_{a.e} \hat{P}(\xi),
\end{aligned}
$$

where equality is almost everywhere and follows from the fact that if $\int f = 0$ then $f = 0$ almost everywhere.

6 Problem 6

We first note that if $G_t(x) \coloneqq t^{-n}e^{-\pi|x|^2/t^2}$, then $\hat{G}_t(\xi) = e^{-\pi t^2|\xi|^2}$.

Moreover, if an interchange of integrals is justified, we have have

$$\begin{aligned}
\|f\|_1 &:= \int_{\mathbb{R}^n} \left| \int_0^\infty G_t(x) e^{-\pi t^2} t^{2\varepsilon-1} \, dt \right| \, dx \\
&= \int_{\mathbb{R}^n} \int_0^\infty G_t(x) e^{-\pi t^2} t^{2\varepsilon-1} \, dt \, dx \\
&\quad \text{since the integrand and thus integral is positive.} \\
&=_? \int_0^\infty \int_{\mathbb{R}^n} G_t(x) e^{-\pi t^2} t^{2\varepsilon-1} \, dx \, dt \\
&= \int_0^\infty e^{-\pi t^2} t^{2\varepsilon-1} \left(\int_{\mathbb{R}^n} G_t(x) \, dx \right) \, dt \\
&= \int_0^\infty e^{-\pi t^2} t^{2\varepsilon-1} \, (1) \, dt \\
&= \int_0^\infty e^{-\pi t^2} t^{2\varepsilon-1} \, dt,
\end{aligned}$$

which we claim is finite, so $f \in L^1$.

To see that the norm is finite, we note that

$$t \in [0,1] \implies e^{-\pi t^2} < 1$$

and if we take $\varepsilon < \frac{1}{2}$, we have $2\varepsilon - 1 < 0$ and thus

$$t \in [1,\infty) \implies t^{2\varepsilon-1} \leq 1.$$

Thus

$$\begin{aligned}
\int_0^\infty e^{-\pi t^2} t^{2\varepsilon-1} \, dt &= \int_0^1 e^{-\pi t^2} t^{2\varepsilon-1} \, dt + \int_1^\infty e^{-\pi t^2} t^{2\varepsilon-1} \, dt \\
&\leq \int_0^1 t^{2\varepsilon-1} \, dt + \int_1^\infty e^{-\pi t^2} \, dt \\
&\leq \int_0^1 t^{2\varepsilon-1} \, dt + \int_0^\infty e^{-\pi t^2} \, dt \\
&= \frac{1}{2\varepsilon} + \frac{1}{2} < \infty,
\end{aligned}$$

where the first term is obtained by directly evaluating the integral, and the second is derived from the fact that its integral over the real line is 1 and it is an even function.

> Justifying the interchange: we note that the integrand $G_t(x)e^{-\pi t^2} t^{2\varepsilon-1}$ is non-negative, and we've just showed that one of the iterated integrals is absolutely convergent, so Tonelli will apply if the integrand is measurable. But $G_t(x)$ is a continuous function on $\mathbb{R}^n$ and the remaining terms are continuous on $\mathbb{R}$, so they are all measurable on $\mathbb{R}^n$ and $\mathbb{R}$ respectively But then taking cylinders on everything in sight yields measurable functions, and the product of measurable functions is measurable.

If another interchange of integrals is justified, we can compute

$$
\begin{aligned}
\hat{f}(\xi) &\coloneqq \int_{\mathbb{R}^n} \left(\int_0^\infty G_t(x) e^{-\pi t^2} t^{2\varepsilon - 1} \, dt \right) e^{-2\pi i x \cdot \xi} \, dx \\
&= \int_{\mathbb{R}^n} \int_0^\infty G_t(x) e^{-\pi t^2} t^{2\varepsilon - 1} e^{-2\pi i x \cdot \xi} \, dt \, dx \\
&=_? \int_0^\infty \int_{\mathbb{R}^n} G_t(x) e^{-\pi t^2} t^{2\varepsilon - 1} e^{-2\pi i x \cdot \xi} \, dx \, dt \\
&= \int_0^\infty e^{-\pi t^2} t^{2\varepsilon - 1} \left(\int_{\mathbb{R}^n} G_t(x) e^{-2\pi i x \cdot \xi} \, dx \right) \, dt \\
&= \int_0^\infty e^{-\pi t^2} t^{2\varepsilon - 1} \hat{G}_t(\xi) \, dt \\
&= \int_0^\infty e^{-\pi t^2} t^{2\varepsilon - 1} e^{-\pi t^2 |\xi|^2} \, dt \\
&= \int_0^\infty e^{-\pi t^2 (1 + |\xi|^2)} t^{2\varepsilon - 1} \, dt \\
&= \int_0^\infty e^{-\pi (t \sqrt{1 + |\xi|^2})^2} t^{2\varepsilon - 1} \, dt \\
&\qquad s = t \sqrt{1 + |\xi|^2}, \; ds = \sqrt{1 + |\xi|^2} dt \\
&= \int_0^\infty e^{-\pi s^2} \left(\frac{s}{\sqrt{1 - |\xi|^2}} \right)^{2\varepsilon - 1} \frac{1}{\sqrt{1 + |\xi|^2}} \, ds \\
&= (1 + |\xi|^2)^{-\frac{2\varepsilon - 1}{2}} (1 + |\xi|^2)^{-\frac{1}{2}} \int_0^\infty e^{-\pi s^2} s^{2\varepsilon - 1} \, ds \\
&= (1 + |\xi|^2)^{-\varepsilon} \int e^{-\pi t^2} t^{2\varepsilon - 1} \, dt \\
&\coloneqq F(\xi) \|f\|_1.
\end{aligned}
$$

To see that the interchange is justified, note that

$$
\int_{\mathbb{R}^n} \int_0^\infty \left| G_t(x) e^{-\pi t^2} t^{2\varepsilon - 1} e^{-2\pi i x \cdot \xi} \right| \, dt \, dx = \int_{\mathbb{R}^n} \int_0^\infty \left| G_t(x) e^{-\pi t^2} t^{2\varepsilon - 1} \right| \, dt \, dx,
$$

since $\left| e^{2\pi i x \cdot \xi} \right| = 1$. The integrand appearing is precisely what we showed was measurable when computed $\|f\|_1$ above, so Tonelli applies.

Thus $F(\xi)$ is the Fourier transform of the function $g(x) \coloneqq f(x)/\|f\|_1$. $\square$

Problem Set 7

D. Zack Garza

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Contents

1 Problem 1

1.1 Part a

We want to show that $\ell^2(\mathbb{N})$ is complete, so let $\{x_n\} \subseteq \ell^2(\mathbb{N})$ be a Cauchy sequence. We then have $\|x^j - x^k\|_{\ell^2} \to 0$, and we want to produce some $\mathbf{x} \coloneqq \lim_{n\to\infty} x^n$ such that $x \in \ell^2$.

To this end, for each fixed index i, define

$$\mathbf{x}_i \coloneqq \lim_{n\to\infty} x_i^n.$$

This is well-defined since $\|x^j - x^k\|_{\ell^2} = \sum_i \left|x_i^j - x_i^k\right|^2 \to 0$, and since this is a sum of positive real numbers that approaches zero, each term must approach zero. But then for a fixed i, the sequence $\left|x_i^j - x_i^k\right|^2$ is a Cauchy sequence of real numbers which necessarily converges by thethe completeness of $\mathbb{R}$.

We also have $\|\mathbf{x} - x^j\|_{\ell^2} \to 0$ since

$$\|\mathbf{x} - x^j\|_{\ell^2} = \left\|\lim_{k\to\infty} x^k - x^j\right\|_{\ell^2} = \lim_{k\to\infty} \|x^k - x^j\|_{\ell^2} \to 0$$

where the limit can be passed through the norm because the map $t \mapsto \|t\|_{\ell^2}$ is continuous. So $x^j \to \mathbf{x}$ in ℓ^2 as well.

It remains to show that $\mathbf{x} \in \ell^2(\mathbb{N})$, i.e. that $\sum_i |\mathbf{x}_i|^2 < \infty$. To this end, we write

$$\begin{aligned}
\|\mathbf{x}\|_{\ell^2} &= \|\mathbf{x} - x^j + x^j\|_{\ell^2} \\
&\leq \|\mathbf{x} - x^j\|_{\ell^2} + \|x^j\|_{\ell^2} \\
&\to M < \infty,
\end{aligned}$$

where $\lim_j \|\mathbf{x} - x^j\|_{\ell^2} = 0$ by the previous argument, and the second term is bounded because $x^j \in \ell^2 \iff \|x^j\|_{\ell^2} \coloneqq M < \infty$. $\square$

1.2 Part b

Let H be a Hilbert space with inner product $\langle\cdot,\ \cdot\rangle$ and induced norm $\|\cdot\|$.

Lemma: For any complex number z, we have

$$\Im(z) = \Re(-iz),$$

and as a corollary, since the inner product on H takes values in $\mathbb{C}$, we have

$$\Re(\langle x,\ iy\rangle) = \Re(-i\langle x,\ y\rangle) = \Im(\langle x,\ y\rangle).$$

We can compute the following:

$$\|x+y\|^2 = \|x\|^2 + \|y\|^2 + 2\,\Re\left(\langle x,\ y\rangle\right)$$

$$\|x-y\|^2 = \|x\|^2 + \|y\|^2 - 2\,\Re\left(\langle x,\ y\rangle\right)$$

$$\begin{aligned}\|x+iy\|^2 &= \|x\|^2 + \|y\|^2 + 2\,\Re\left(\langle x,\ iy\rangle\right)\\ &= \|x\|^2 + \|y\|^2 + \Im(\langle x,\ y\rangle)\end{aligned}$$

$$\begin{aligned}\|x-iy\|^2 &= \|x\|^2 + \|y\|^2 - 2\,\Re\left(\langle x,\ iy\rangle\right)\\ &= \|x\|^2 + \|y\|^2 + \Im(\langle x,\ y\rangle)\end{aligned}$$

and summing these all

$$\begin{aligned}\|x+y\|^2 - \|x-y\|^2 + i\|x+iy\|^2 - i\|x+iy\| &= 4\,\Re\left(\langle x,\ y\rangle\right) + 4i\,\Im\left(\langle x,\ y\rangle\right)\\ &= 4\langle x,\ y\rangle.\end{aligned}$$

To conclude that a linear map U is an isometry iff U is unitary, if we assume U is unitary then we can write

$$\|x\|^2 \coloneqq \langle x,\ x\rangle = \langle Ux,\ Ux\rangle \coloneqq \|Ux\|^2.$$

Assuming now that U is an isometry, by the polarization identity we can write

$$\begin{aligned}\langle Ux,\ Uy\rangle &= \frac{1}{4}\left(\|Ux+Uy\|^2 + \|Ux-Uy\|^2 + i\|Ux+Uy\|^2 - i\|Ux+Uy\|^2\right)\\ &= \frac{1}{4}\left(\|U(x+y)\|^2 + \|U(x-y)\|^2 + i\|U(x+y)\|^2 - i\|U(x+y)\|^2\right)\\ &= \frac{1}{4}\left(\|x+y\|^2 + \|x-y\|^2 + i\|x+y\|^2 - i\|x+y\|^2\right)\\ &= \langle x,\ y\rangle.\end{aligned}$$

□

2 Problem 2

Lemma: The map $\langle\,\cdot\,,\ \cdot\,\rangle : H \times H \to \mathbb{R}$ is continuous.

Proof:

Let $x_n \to x$ and $y_n \to y$, then

$$
\begin{aligned}
|\langle x_n,\ y_n\rangle - \langle x,\ y\rangle| &= |\langle x_n,\ y_n\rangle - \langle x,\ y_n\rangle + \langle x,\ y_n\rangle - \langle x,\ y\rangle| \\
&= |\langle x_n - x,\ y_n\rangle + \langle x,\ y_n - y\rangle| \\
&\le \|x_n - x\|\|y_n\| + \|x\|\|y_n - y\| \\
&\to 0 \cdot M + C \cdot 0 < \infty,
\end{aligned}
$$

where $\|y_n\| \to \|y\| \coloneqq M < \infty$ since $y \in H$ implies that $\|y\|$ is finite.

2.1 Part a:

We want to show that sequences in $E^\perp$ converge to elements of $E^\perp$. Using the lemma, letting $\{e_n\}$ be a sequence in $E^\perp$, so $y \in E \implies \langle e_n,\ y\rangle = 0$. Since H is complete, $e_n \to e \in H$; we can show that $e \in E^\perp$ by letting $y \in E$ be arbitrary and computing

$$
\langle e,\ y\rangle = \left\langle \lim_n e_n,\ y\right\rangle = \lim_n \langle e_n,\ y\rangle = \lim_n 0 = 0,
$$

so $e \in E^\perp$.

2.2 Part b:

Let $S \coloneqq \operatorname{span}_H(E)$; then the smallest closed subspace containing E is $\overline{S}$, the closure of S. We will proceed by showing that $E^{\perp\perp} = \overline{S}$.

$\overline{S} \subseteq E^{\perp\perp}$:

Let $\{x_n\}$ be a sequence in S, so $x_n \to x \in \overline{S}$.

First, each x_n is in $E^{\perp\perp}$, since if we write $x_n = \sum a_i e_i$ where $e_i \in E$, we have

$$
y \in E^\perp \implies \langle x_n,\ y\rangle = \left\langle \sum_i a_i e_i,\ y\right\rangle = \sum_i a_i \langle e_i,\ y\rangle = 0 \implies x_n \in (E^\perp)^\perp.
$$

It remains to show that $x \in E^{\perp\perp}$, which follows from

$$
y \in E^\perp \implies \langle x,\ y\rangle = \left\langle \lim_n x_n,\ y\right\rangle = \lim_n \langle x_n,\ y\rangle = 0 \implies x \in (E^\perp)^\perp,
$$

where we've used continuity of the inner product.

$E^{\perp\perp} \subseteq \overline{S}$:

For notational convenience, let S_c denote the closure $\overline{S}$. Let $x \in E^{\perp\perp}$. Noting that S_c is closed, we can define P, the operator projecting elements onto S_c, and write

$$
x = Px + (x - Px) \in S_c \oplus S_c^\perp
$$

But since $\langle x,\ x - Px\rangle = 0$ (because $x - Px \in E^\perp$ and $x \in (E^\perp)^\perp$), we can rewrite the first term in this inner product to obtain

$$0 = \langle x,\ x - Px\rangle = \langle Px + (x - Px),\ x - Px\rangle = \langle Px,\ x - Px\rangle + \langle x - Px,\ x - Px\rangle,$$

where we can note that the first term is zero because $Px \in S_c$ and $x - Px \in S_c^\perp$, and the second term is $\|x - Px\|^2$.

But this says $\|x - Px\|^2 = 0$, so $x - Px = 0$ and thus $x = Px \in S_c$, which is what we wanted to show.

3 Problem 3

3.1 Part a

We compute

$$\begin{aligned}
\|e_0\|^2 &= \int_0^1 1^2\ dx = 1 \\
\|e_1\|^2 &= \int_0^1 3(2x-1)^2 = \frac{1}{2}(2x-1)^2\Big|_0^1 = 1 \\
\langle e_0,\ e_1\rangle &= \int_0^1 \sqrt{3}(2x-1)\ dx = \frac{\sqrt{3}}{4}(2x-1)\Big|_0^1 = 0.
\end{aligned}$$

which verifies that this is an orthonormal system.

3.2 Part b

We first note that this system spans the degree 1 polynomials in $L^2([0,1])$, since we have

$$\begin{bmatrix} 1 & 0 \\ 2\sqrt{3} & \sqrt{3} \end{bmatrix} [1, x]^t = [e_0, e_1]$$

which exhibits a matrix that changes basis from $\{1, x\}$ to $\{e_0, e_1\}$ which is invertible, so both sets span the same subspace.

Thus the closest degree 1 polynomial f to x^3 is given by the projection onto this subspace, and since $\{e_i\}$ is orthonormal this is given by

$$
\begin{aligned}
f(x) &= \sum_i \Big\langle x^3,\ e_i \Big\rangle e_i \\
&= \Big\langle x^3,\ 1 \Big\rangle 1 + \Big\langle x^3,\ \sqrt{3}(2x-1) \Big\rangle \sqrt{3}(2x-1) \\
&= \int_0^1 x^2\ dx + \sqrt{3}(2x-1) \int_0^1 \sqrt{3}x^2(2x-1)\ dx \\
&= \frac{1}{3} + \sqrt{3}(2x-1)\frac{\sqrt{3}}{6} \\
&= x - \frac{1}{6}.
\end{aligned}
$$

We can also compute

$$
\begin{aligned}
\|f-g\|_2^2 &= \int_0^1 (x^2 - x + \frac{1}{6})^2\ dx \\
&= \frac{1}{180} \\
\implies \|f-g\|_2 &= \frac{1}{\sqrt{180}}.
\end{aligned}
$$

4 Problem 4

4.1 Part a

4.1.1 i

We can first note that $\Big\langle 1/\sqrt{2},\ \cos(2\pi nx) \Big\rangle = \Big\langle 1/\sqrt{2},\ \sin(2\pi mx) \Big\rangle = 0$ for any n or m, since this involves integrating either sine or cosine over an integer multiple of its period.

Letting $m, n \in \mathbb{Z}$, we can then compute

$$
\begin{aligned}
\langle \cos(2\pi nx),\ \sin(2\pi mx) \rangle &= \int_0^1 \cos(2\pi nx)\sin(2\pi mx)\ dx \\
&= \frac{1}{2}\int_0^1 \sin(2\pi(n+m)x) - \sin(2\pi(n-m)x)\ dx \\
&= \frac{1}{2}\int_0^1 \sin(2\pi(n+m)x) - \frac{1}{2}\int_0^1 \sin(2\pi(n-m)x)\ dx \\
&= 0,
\end{aligned}
$$

which again follows from integration of sine over a multiple of its period (where we use the fact that $m+n, m-n \in \mathbb{Z}$).

Similarly,

$$
\begin{aligned}
\langle \cos(2\pi nx),\ \cos(2\pi mx)\rangle &= \int_0^1 \cos(2\pi nx)\cos(2\pi mx)\ dx \\
&= \frac{1}{2}\int_0^1 \cos(2\pi(m+n)x) + \cos(2\pi(m-n)x)\ dx \\
&= \begin{cases} \frac{1}{2}\int_0^1 \cos(4\pi nx) + 1\ dx = 1 & m = n \\ 0 & m \neq n \end{cases}.
\end{aligned}
$$

$$
\begin{aligned}
\langle \sin(2\pi nx),\ \sin(2\pi mx)\rangle &= \int_0^1 \sin(2\pi nx)\sin(2\pi mx)\ dx \\
&= \frac{1}{2}\int_0^1 \cos(2\pi(m-n)x) + \cos(2\pi(m+n)x)\ dx \\
&= \begin{cases} \frac{1}{2}\int_0^1 1 + \cos(4\pi nx)\ dx = 1 & m = n \\ 0 & m \neq n \end{cases}.
\end{aligned}
$$

Thus each pairwise combination of elements are orthonormal, making the entire set orthonormal.

4.1.2 ii

We have

$$
\begin{aligned}
\left\langle e^{2\pi kx},\ e^{-2\pi i\ell x}\right\rangle &= \int_0^1 e^{2\pi ikx}\overline{e^{2\pi i\ell x}}\ dx \\
&= \int_0^1 e^{2\pi ikx}e^{-2\pi i\ell x}\ dx \\
&= \int_0^1 e^{2\pi i(k-\ell)x}\ dx \\
(&= \int_0^1 1\ dx = 1 \quad \text{if } k = \ell, \text{ otherwise:}) \\
&= \frac{e^{2\pi i(k-\ell)x}}{2\pi i(k-\ell)}\Big|_0^1 \\
&= \frac{e^{2\pi i(k-\ell)} - 1}{2\pi i(k-\ell)} \\
&= 0,
\end{aligned}
$$

since $e^{2\pi ik} = 1$ for every $k \in Z$, and $k - \ell \in \mathbb{Z}$. Thus this set is orthonormal.

4.2 Part b

4.2.1 i

By the Weierstrass approximation theorem for functions on a bounded interval, we can find a polynomials $P_n(x)$ such that $\|f - P_n\|_\infty \to 0$, i.e. the P_n uniformly approximate f on $[0, 1]$.

Letting $\varepsilon > 0$, we can thus choose a P such that $\|f - P\|_\infty < \varepsilon$, which necessarily implies that $\|f - P\|_{L^1} < \varepsilon$ since we have

$$\int_0^1 |f(x) - P(x)| \, dx \le \int_0^1 \varepsilon \, dx = \varepsilon.$$

Thus we can write

$$f(x) = P(x) + (f(x) - P(x))$$

where $h(x) \coloneqq f(x) - P(x)$ satisfies $\|h\|_{L^1} < \varepsilon$. It only remains to show that $P \in L^2([0,1])$, but this follows from the fact that any polynomial on a compact interval is uniformly bounded, say $|P(x)| \le M < \infty$ for all $x \in [0,1]$, and thus

$$\|P\|_{L^2}^2 = \int_0^1 |P(x)|^2 \, dx \le \int_0^1 M^2 \, dx = M^2 < \infty.$$

It follows that we can let $g = P$ and $h = f - P$ to obtain the desired result.

4.2.2 ii

By part (i), the claim is that it suffices to show this is true for $f \in L^2$. In this case, we can identify

$$\begin{aligned} \int_0^1 f(x) \cos(2\pi k x) \, dx &\coloneqq \Re(\hat{f}(k)) \\ \int_0^1 f(x) \sin(2\pi k x) \, dx &\coloneqq \Im(\hat{f}(k)), \end{aligned}$$

the real and imaginary parts of the kth Fourier coefficient of f respectively.

By Bessel's inequality, we know that $\left\{\hat{f}(k)\right\}_{k\in\mathbb{N}} \in \ell^1(\mathbb{N})$, and so $\sum_k \left|\hat{f}(k)\right| < \infty$.

But this is a convergent sequence of real numbers, which necessarily implies that $\left|\hat{f}(k)\right| \to 0$. In particular, this also means that its real and imaginary parts tend to zero, which is exactly what we wanted to show.

If we instead have $f \in L^1$, write $f = g + h$ where $g \in L^2$ and $\|h\|_{L^1} \to 0$. Then

$$\begin{aligned} \left|\int_0^1 f(x) \cos(2\pi k x) \, dx\right| &= \left|\int_0^1 (g(x) + h(x)) \cos(2\pi k x) \, dx\right| \\ &\le \left|\int_0^1 g(x) \cos(2\pi k x) \, dx\right| + \left|\int_0^1 h(x) \cos(2\pi k x) \, dx\right| \\ &\le \left|\int_0^1 g(x) \cos(2\pi k x) \, dx\right| + \int_0^1 |h(x)||\cos(2\pi k x)| \, dx \\ &= |\hat{g}(k)| + \varepsilon \\ &\to 0, \end{aligned}$$

with a similar computation for $\int f(x) \sin(2\pi k x)$. $\square$

5 Problem 5

5.1 Part 1

We use the following algorithm: given $\{v\}_i$, we set

- $e_1 = v_1$, and then normalize to obtain $\hat{e_1} = e_1/\|e_1\|$
- $e_i = v_i - \sum_{k \leq i-1} \langle v_i,\ \hat{e_i} \rangle \hat{e_i}$

The result set $\{\hat{e_i}\}$ is the orthonormalized basis.

We set $e_1 = 1$, and check that $\|e_1\|^2 = 2$, and thus set $\hat{e}_1 = \frac{1}{\sqrt{2}}$.

We then set

$$\begin{aligned} e_2 &= x - \langle x,\ \hat{e}_1 \rangle \hat{e}_1 \\ &= x - \langle x,\ 1 \rangle 1 \\ &= x - \int_{-1}^{1} \frac{1}{\sqrt{2}} x\ dx \\ &= x - \int \text{odd function} \\ &= x, \end{aligned}$$

and so $e_2 = x$. We can then check that

$$\|e_2\| = \left(\int_{-1}^{1} x^2\ dx \right)^{1/2} = \sqrt{\frac{2}{3}},$$

and so we set $\hat{e}_2 = \sqrt{\frac{3}{2}} x$.

We continue to compute

$$\begin{aligned} e_3 &= x^2 - \left\langle x^2,\ \hat{e}_1 \right\rangle \hat{e}_1 - \left\langle x^2,\ \hat{e}_2 \right\rangle \hat{e}_2 \\ &= x^2 - \frac{1}{2} \int_{-1}^{1} x^2\ dx - \frac{3}{2} x \int_{-1}^{1} x^3\ dx \\ &= x^2 - \left(\frac{1}{6} x^3 \right) |_{-1}^{1} + \frac{3}{2} x \int_{-1}^{1} \text{odd function} \\ &= x^2 - \frac{1}{3}. \end{aligned}$$

We can then check that $\|e_3\|^2 = \frac{8}{45}$, so we set

$$
\begin{aligned}
\hat{e}_3 &= \sqrt{\frac{45}{8}}(x^2 - \frac{1}{3}) \\
&= \frac{1}{2}\sqrt{\frac{45}{2}}\frac{1}{3}(3x^2 - 1) \\
&= \frac{1}{3}\sqrt{\frac{45}{2}}\left(\frac{3x^2 - 1}{2}\right).
\end{aligned}
$$

In summary, this yields

$$
\begin{aligned}
\hat{e}_1 &= \frac{1}{\sqrt{2}} \\
\hat{e}_2 &= x \\
\hat{e}_3 &= \frac{1}{3}\sqrt{\frac{45}{2}}\left(\frac{3x^2 - 1}{2}\right),
\end{aligned}
$$

which are scalar multiples of the first three Legendre polynomials.

5.2 Part b

Let $p(x) = a + bx + cx^2$, we are then looking for p such that $\|x^3 - p(x)\|_2^2$ is minimized. Noting that

$$
p(x) \in \operatorname{span}\left\{1, x, x^2\right\} = \operatorname{span}\left\{P_0(x), P_1(x), P_2(x)\right\} := S,
$$

we can conclude that $p(x)$ will be the projection of x^3 onto S. Thus $p(x) = \sum_{i=0}^{2} \langle x^3,\ \hat{e}_i\rangle \hat{e}_i$.

Proceeding to compute the terms in this expansion, we can note that $\langle x^3,\ f\rangle$ for any f that is even will result in integrating an odd function over a symmetric interval, yielding zero. So only one term doesn't vanish:

$$
\left\langle x^3,\ x\right\rangle x = x\int_{-1}^{1} x^4\ dx = \frac{2}{5}x
$$

.

And thus $p(x) = \frac{2}{5}x$ is the minimizer.

5.3 Part c

The first three conditions necessitate $g \in S^{\perp}$ and $\|g\| = 1$. Since S is a closed subspace, we can write $x^3 = p(x) + (x^3 - p(x)) \in S \oplus S^{\perp}$, and so $x^3 - p(x) \in S^{\perp}$.

The claim is that $g(x) := x^3 - p(x)$ is a scalar multiple of the desired maximizer. This follows from the fact that

$$
\left|\left\langle x^3 - p,\ g\right\rangle\right| \leq \|x^3 - p\|\|g\|
$$

by Cauchy-Schwarz, with equality precisely when $g = \lambda(x^3 - p)$ for some scalar λ. However, the restriction $\|g\| = 1$ forces $\lambda = \|x^3 - p\|^{-1}$.

A computation shows that

$$\|x^3 - p\|^2 = \int_0^1 (x^3 - \frac{2}{5}x)^2 \, dx = \frac{19}{525},$$

and so we can take

$$g(x) := \frac{25}{\sqrt{19}} \left(x^3 - \frac{2}{5}x \right).$$

6 Problem 6

6.1 Part a

To see that $g \in \mathcal{C}$, we can compute

$$\begin{aligned} \langle g, \ 1 \rangle &= \int_0^1 18x^2 - 5 \, dx = 6 - 5 = 1 \\ \langle g, \ x \rangle &= \int_0^1 18x^3 - 5x \, dx = \frac{18}{4} - \frac{5}{2} = 2. \end{aligned}$$

To see that $\mathcal{C} = g + S^{\perp}$, let $f \in \mathcal{C}$, so $\langle f, \ 1 \rangle = 1$ and $\langle f, \ x \rangle = 2$. We can then conclude that $f - g \in S^{\perp}$, since we have

$$\begin{aligned} \langle f - g, \ 1 \rangle &= \langle f, \ 1 \rangle - \langle g, \ 1 \rangle = 1 - 1 = 0 \\ \langle f - g, \ x \rangle &= \langle f, \ x \rangle - \langle g, \ x \rangle = 2 - 2 = 0. \end{aligned}$$

6.2 Part b

Note that this equivalent to finding an $f_0 \in \mathcal{C}$ such that $\|f_0\|$ is minimized.

Letting $f_0 \in \mathcal{C}$, be arbitrary and noting that by part (a) we have $f_0 = g + s$ where $s \in S^{\perp}$, we can compute

$$\begin{aligned} \|f_0\|^2 &= \langle f_0, \ f_0 \rangle \\ &= \langle g + s, \ g + s \rangle \\ &= \|g\|^2 + 2\Re\langle g, \ s \rangle + \|s\|^2, \end{aligned}$$

which can be minimized by taking $s = 0$, which forces $\|s\|^2 = 0$ and $\langle g, \ s \rangle = 0$. But this imposes the condition $f_0 = g + 0 = g$. $\square$

Problem Set 8

D. Zack Garza

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Contents

1 Problem 1

1.1 Part a

It follows from the definition that $\|f\|_\infty = 0 \iff f = 0$ almost everywhere, and if $\|f\|_\infty$ is the best upper bound for f almost everywhere, then $\|cf\|_\infty$ is the best upper bound for cf almost everywhere.

So it remains to show the triangle inequality. Suppose that $|f(x)| \leq \|f\|_\infty$ a.e. and $|g(x)| \leq \|g\|_\infty$ a.e., then by the triangle inequality for the $|\cdot|_{\mathbb{R}}$ we have

$$\begin{aligned}|(f+g)(x)| &\leq |f(x)| + |g(x)| \quad a.e. \\ &\leq \|f\|_\infty + \|g\|_\infty \quad a.e.,\end{aligned}$$

which means that $\|f+g\|_\infty \leq \|f\|_\infty + \|g\|_\infty$ as desired.

1.2 Part b

$\implies$: Suppose $\|f_n - f\|_\infty \to 0$, then for every ε, N_ε can be chosen large enough such that $|f_n(x) - f(x)| < \varepsilon$ a.e., which precisely means that there exist sets E_ε such that $x \in E_\varepsilon \implies |f_n(x) - f(x)|$ and $m(E_\varepsilon^c) = 0$.

But then taking the sequence $\varepsilon_n := \frac{1}{n} \to 0$, we have $f_n \rightrightarrows f$ uniformly on $E := \bigcap_n E_n$ by definition, and $E^c = \bigcup_n E_n^c$ is still a null set.

$\impliedby$: Suppose $f_n \rightrightarrows f$ uniformly on some set E and $m(E^c) = 0$. Then for any ε, we can choose N large enough such that $|f_n(x) - f(x)| < \varepsilon$ on E; but then ε is an upper bound for $f_n - f$ almost everywhere, so $\|f_n - f\|_\infty < \varepsilon \to 0$.

1.3 Part c

To see that simple functions are dense in $L^\infty(X)$, we can use the fact that $f \in L^\infty(X) \iff$ there exists a g such that $f = g$ a.e. and g is bounded.

Then there is a sequence s_n of simple functions such that $\|s_n - g\|_\infty \to 0$, which follows from a proof in Folland:

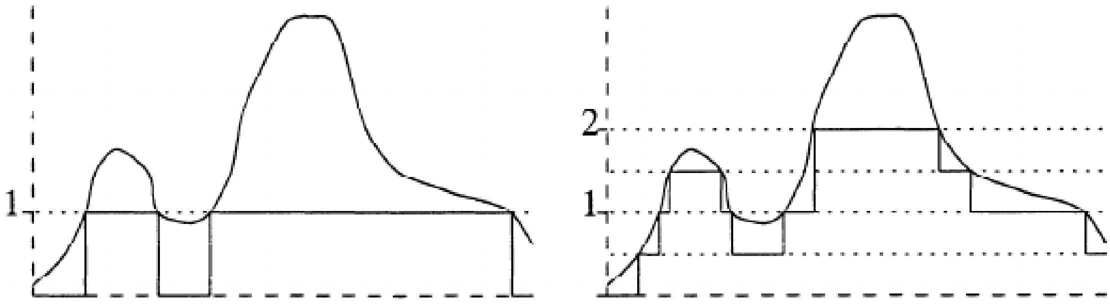
Proof. (a) For $n = 0, 1, 2, \ldots$ and $0 \leq k \leq 2^{2n} - 1$, let

$$E_n^k = f^{-1}\big((k2^{-n},\,(k+1)2^{-n}]\big) \quad \text{and} \quad F_n = f^{-1}\big((2^n, \infty]\big),$$

and define

$$\phi_n = \sum_{k=0}^{2^{2n}-1} k2^{-n}\chi_{E_n^k} + 2^n\chi_{F_n}.$$

(This formula is messy in print but easily understood graphically; see Figure 2.1.) It is easily checked that $\phi_n \leq \phi_{n+1}$ for all n, and $0 \leq f - \phi_n \leq 2^{-n}$ on the set where $f \leq 2^n$. The result therefore follows.



However, $C_c^0(X)$ is dense $L^\infty(X) \iff$ every $f \in L^\infty(X)$ can be approximated by a sequence $\{g_k\} \subset C_c^0(X)$ in the sense that $\|f - g_n\|_\infty \to 0$. To see why this can *not* be the case, let $f(x) = 1$, so $\|f\|_\infty = 1$ and let $g_n \to f$ be an arbitrary sequence of C_c^0 functions converging to f pointwise.

Since every g_n has compact support, say $\operatorname{supp}(g_n) \coloneqq E_n$, then $g_n|_{E_n^c} \equiv 0$ and $m(E_n^c) > 0$. In particular, this means that $\|f - g_n\|_\infty = 1$ for every n, so g_n can not converge to f in the infinity norm.

2 Problem 2

2.1 Part a

2.1.1 Part i

Lemma: $\|1\|_p = m(X)^{1/p}$

This follows from $\|1\|_p^p = \int_X |1|^p = \int_X 1 = m(X)$ and taking pth roots. $\square$

By Holder with $p = q = 2$, we can now write

$$\begin{aligned} \|f\|_1 &= \|1 \cdot f\|_1 \le \|1\|_2 \|f\|_2 = m(X)^{1/2} \|f\|_2 \\ \implies \|f\|_1 &\le m(X)^{1/2} \|f\|_2. \end{aligned}$$

Letting $M \coloneqq \|f\|_\infty$, We also have

$$\begin{aligned} \|f\|_2^2 &= \int_X |f|^2 \le \int_X |M|^2 = M^2 \int_X 1 = M^2 m(X) \\ \implies \|f\|_2 &\le m(X)^{1/2} \|f\|_\infty \\ \implies m(X)^{1/2} \|f\|_2 &\le m(X) \|f\|_\infty, \end{aligned}$$

and combining these yields

$$\|f\|_1 \le m(X)^{1/2} \|f\|_2 \le m(X) \|f\|_\infty,$$

from which it immediately follows

$$m(X) < \infty \implies L^\infty(X) \subseteq L^2(X) \subseteq L^1(X).$$

The Inclusions Are Strict:

1. $\exists f \in L^1(X) \setminus L^2(X)$:

Let $X = [0,1]$ and consider $f(x) = x^{-\frac{1}{2}}$. Then

$$\|f\|_1 = \int_0^1 x^{-\frac{1}{2}} < \infty \qquad \text{by the } p \text{ test},$$

while

$$\|f\|_2^2 = \int_0^1 x^{-1} \to \infty \qquad \text{by the } p \text{ test}.$$

2. $\exists f \in L^2(X) \setminus L^\infty(X)$:

Take $X = [0,1]$ and $f(x) = x^{-\frac{1}{4}}$. Then

$$\|f\|_2^2 = \int_0^1 x^{-\frac{1}{4}} < \infty \qquad \text{by the } p \text{ test},$$

while $\|f\|_\infty > M$ for any finite M, since f is unbounded in neighborhoods of 0, so $\|f\|_\infty = \infty$.

2.1.2 Part ii

1. $\exists f \in L^2(X) \setminus L^1(X)$ when $m(X) = \infty$:

Take $X = [1, \infty)$ and let $f(x) = x^{-1}$, then

$$\|f\|_2^2 = \int_0^\infty x^{-2} < \infty \qquad \text{by the } p \text{ test},$$
$$\|f\|_1 = \int_0^\infty x^{-1} \to \infty \qquad \text{by the } p \text{ test}.$$

2. $\exists f \in L^\infty(X) \setminus L^2(X)$ when $m(X) = \infty$:

Take $X = \mathbb{R}$ and $f(x) = 1$. then

$$\|f\|_\infty = 1$$
$$\|f\|_2^2 = \int_{\mathbb{R}} 1 \to \infty.$$

3. $L^2(X) \subseteq L^1(X) \implies m(X) < \infty$:

Let $f = \chi_X$, by assumption we can find a constant M such that $\|\chi_X\|_2 \leq M \|\chi_X\|_1$.

Then pick a sequence of sets $E_k \nearrow X$ such that $m(E_k) < \infty$ for all k, $\chi_{E_k} \nearrow \chi_X$, and thus $\|\chi_{E_k}\|_p \leq M \|\chi_E\|_p$. By the lemma, $\|\chi_{E_k}\|_p = m(E_k)^{1/p}$, so we have

$$\begin{aligned}
\|\chi_{E_k}\|_2 \leq M \|\chi_{E_k}\|_1 &\implies \frac{\|\chi_{E_k}\|_2}{\|\chi_{E_k}\|_1} \leq M \\
&\implies \frac{m(E_k)^{1/2}}{m(E_k)} \leq M \\
&\implies m(E_k)^{-1/2} \leq M \\
&\implies m(E_k) \leq M^2 < \infty.
\end{aligned}$$

and by continuity of measure, we have $\lim_K m(E_k) = m(X) \leq M^2 < \infty$. $\square$

2.2 Part b

1. $L_1(X) \bigcap L^\infty(X) \subset L^2(X)$:

Let $f \in L^1(X) \bigcap L^\infty(X)$ and $M := \|f\|_\infty$, then

$$\|f\|_2^2 = \int_X |f|^2 = \int_X |f||f| \leq \int_X M|f| = M \int |f| := \|f\|_\infty \|f\|_1 < \infty. \tag{1}$$

The inclusion is strict, since we know from above that there is a function in $L^2(X)$ that is not in $L^\infty(X)$.

Note that taking square roots in (1) immediately yields

$$\|f\|_{L^2(X)} \leq \|f\|_{L^1(X)}^{1/2} \|f\|_{L^\infty(X)}^{1/2}.$$

2. $L^2(X) \subset L^1(X) + L^\infty(X)$:

Let $f \in L^2(X)$, then write $S = \{x \ni |f(x)| \geq 1\}$ and $f = \chi_S f + \chi_{S^c} f := g + h$.

Since $x \geq 1 \implies x^2 \geq x$, we have

$$\|g\|_1^2 = \int_X |g| = \int_S |f| \leq \int_S |f|^2 \leq \int_X |f|^2 = \|f\|_2^2 < \infty,$$

and so $g \in L^1(X)$.

To see that $h \in L^\infty(X)$, we just note that h is bounded by 1 by construction, and so $\|h\|_\infty \leq 1 < \infty$.

3 Problem 3

For notational convenience, it suffices to prove this for $\ell^p(\mathbb{N})$, where we re-index each sequence in $\ell^p(\mathbb{Z})$ using a bijection $\mathbb{Z} \to \mathbb{N}$.

> Note: this technically reorders all sums appearing, but since we are assuming absolute convergence everywhere, this can be done. One can also just replace $\sum_{j=n}^m |a_j|^p$ with $\sum_{n \leq |j| \leq m} |a_j|^p$ in what follows.

1. $\ell^1(\mathbb{N}) \subset \ell^2(\mathbb{N})$:

Suppose $\sum_j |a|_j < \infty$, then its tails go to zero, so choose N large enough so that

$$j \geq N \implies |a_j| < 1.$$

But then

$$j \geq N \implies |a_j|^2 < |a_j|,$$

and

$$\begin{aligned}
\sum_j |a_j|^2 &= \sum_{j=1}^{N} |a_j|^2 + \sum_{j=N+1}^{\infty} |a_j|^2 \\
&\leq \sum_{j=1}^{N} |a_j|^2 + \sum_{j=N+1}^{\infty} |a_j| \\
&\leq M + \sum_{j=N+1}^{\infty} |a_j| \\
&\leq M + \sum_{j=1}^{\infty} |a_j| \\
&< \infty.
\end{aligned}$$

where we just note that the first portion of the sum is a finite sum of finite numbers and thus bounded.

To see that the inclusion is strict, take $\mathbf{a} \coloneqq \{j^{-1}\}_{j=1}^{\infty}$; then $\|\mathbf{a}\|_2 < \infty$ by the p-test by $\|\mathbf{a}\|_1 = \infty$ since it yields the harmonic series.

2. $\ell^2(\mathbb{N}) \subset \ell^\infty(\mathbb{N})$:

This follows from the contrapositive: if $\mathbf{a}$ is a sequence with unbounded terms, then $\|\mathbf{a}\|_2 = \sum |a_j|^2$ can not be finite, since convergence would require that $|a_j|^2 \to 0$ and thus $|a_j| \to 0$.

To see that the inclusion is strict, take $\mathbf{a} = \{1\}_{j=1}^{\infty}$. Then $\|\mathbf{a}\|_\infty = 1$, but the corresponding sum does not converge.

3. $\|\mathbf{a}\|_2 \leq \|\mathbf{a}\|_1$:

Let $M = \|\mathbf{a}\|_1$, then

$$\|\mathbf{a}\|_2^2 \leq \|\mathbf{a}\|_1^2 \iff \frac{\|\mathbf{a}\|_2^2}{M^2} \leq 1 \iff \sum_j \left|\frac{a_j}{M}\right|^2 \leq 1.$$

But then we can use the fact that

$$\left|\frac{a_j}{M}\right| \leq 1 \implies \left|\frac{a_j}{M}\right|^2 \leq \left|\frac{a_j}{M}\right|$$

to obtain

$$\sum_j \left|\frac{a_j}{M}\right|^2 \leq \sum_j \left|\frac{a_j}{M}\right| = \frac{1}{M}\sum_j |a_j| := 1.$$

4. $\|\mathbf{a}\|_\infty \leq \|\mathbf{a}\|_2$:

This follows from the fact that, we have

$$\|\mathbf{a}\|_\infty^2 := \left(\sup_j |a_j|\right)^2 = \sup_j |a_j|^2 \leq \sum_j |a_j|^2 = \|\mathbf{a}\|_2^2$$

and taking square roots yields the desired inequality.

> Note: the middle inequality follows from the fact that the supremum S is the least upper bound of all of the a_j, so for all j, we have $a_j + \varepsilon > S$ for every $\varepsilon > 0$. But in particular, $a_k + a_j > a_j$ for any pair a_j, a_k where $a_k \neq 0$, so $a_k + a_j > S$ and thus so is the entire sum.

4 Problem 4

4.1 Part a

Let $\{f_k\}$ be a Cauchy sequence, then $\|f_k - f_j\|_u \to 0$. Define a candidate limit by fixing x, then using the fact that $|f_j(x) - f_k(x)| \to 0$ as a Cauchy sequence in $\mathbb{R}$, which converges to some $f(x)$.

We want to show that and $\|f_n - f\|_u \to 0$ and $f \in C([0,1])$.

This is immediate though, since $f_n \to f$ uniformly by construction, and the uniform limit of continuous functions is continuous.

4.2 Part b

It suffices to produce a Cauchy sequence of continuous functions f_k such that $\|f_j - f_j\|_1 \to 0$ but if we define $f(x) := \lim f_k(x)$, we have either $\|f\|_1 = \infty$ or f is not continuous.

To this end, take $f_k(x) = x^k$ for $k = 1, 2, \cdots, \infty$.

Then pointwise we have

$$f_k \to \begin{cases} 0 & x \in [0,1) \\ 1 & x = 1 \end{cases},$$

which has a clear discontinuity, but

$$\|f_k - f_j\|_1 := \int_0^1 x^k - x^j = \frac{1}{k+1} - \frac{1}{j+1} \to 0.$$

5 Problem 5

5.1 Part a

$\Longleftarrow$: It suffices to show that the map

$$\begin{aligned}
&H \twoheadrightarrow \ell^2(\mathbb{N}) \\
&\mathbf{x} \mapsto \{\langle \mathbf{x},\ \mathbf{u}_n\rangle\}_{n=1}^{\infty} \coloneqq \{a_n\}_{n=1}^{\infty}
\end{aligned}$$

is a surjection, and for every $\mathbf{a} \in \ell^2(\mathbb{N})$, we can pull back to some $\mathbf{x} \in H$ such that $\|\mathbf{x}\|_H = \|\mathbf{a}\|_{\ell^2(\mathbb{N})}$.

Following the proof in Neil's notes, let $\mathbf{a} \in \ell^2(\mathbb{N})$ be given by $\mathbf{a} = \{a_j\}$, and define $S_N = \sum_{n=1}^{N} a_n \mathbf{u}_n$. We then have

$$\begin{aligned}
\|S_N - S_M\|_H &= \left\| \sum_{n=M+1}^{N} a_n \mathbf{u}_n \right\|_H \\
&= \sum_{n=M+1}^{N} \|a_n \mathbf{u}_n\|_H && \text{by Pythagoras, since the } \mathbf{u}_n \text{ are orthogonal} \\
&= \sum_{n=M+1}^{N} |a_n|_{\mathbb{C}}\ \|\mathbf{u}_n\|_H \\
&= \sum_{n=M+1}^{N} |a_n|_{\mathbb{C}} && \text{since the } \mathbf{u}_n \text{ are orthonormal} \\
&\to 0 && \text{as } N, M \to \infty,
\end{aligned}$$

which goes to zero because it is the tail of a convergent sum in $\mathbb{R}$.

Since H is complete, every Cauchy sequence converges, and in particular $S_N \to \mathbf{x} \in H$ for some $\mathbf{x}$.

We now have

$$\begin{aligned}
|\langle \mathbf{x},\ \mathbf{u}_n\rangle| &= |\langle \mathbf{x} - S_N + S_N,\ \mathbf{u}_n\rangle| && \forall n, N \\
&= |\langle \mathbf{x} - S_N,\ \mathbf{u}_n\rangle + \langle S_N,\ \mathbf{u}_n\rangle| && \forall n, N \\
&\leq \|\mathbf{x} - S_N\|_H \|\mathbf{u}_n\|_H + |\langle S_N,\ \mathbf{u}_n\rangle| && \forall n, N \text{ by Cauchy-Schwartz} \\
&= \|\mathbf{x} - S_N\|_H + |\langle S_N,\ \mathbf{u}_n\rangle| && \forall n, N \\
&= \|\mathbf{x} - S_N\|_H + |a_n| && \forall N \geq n \\
&\to 0 + |a_n| && \text{as } N \to \infty,
\end{aligned}$$

where we just note that

$$\langle S_N,\ \mathbf{u}_n\rangle = \left\langle \sum_{j=1}^{N} a_j \mathbf{u}_j,\ \mathbf{u}_n \right\rangle = \sum_{j=1}^{N} a_j \langle \mathbf{u}_j,\ \mathbf{u}_n\rangle = a_n \iff N \geq n$$

since $\langle \mathbf{u}_j,\ \mathbf{u}_n\rangle = \delta_{j,n}$ and so the a_n term is extracted iff $\mathbf{u}_n$ actually appears as a summand.

We thus have

$$\langle \mathbf{x},\ \mathbf{u}_n\rangle = |a_n| \quad \forall n,$$

and since $\{\mathbf{u}_n\}$ is a basis, we can apply Parseval's identity to obtain

$$\|\mathbf{x}\|_H^2 = \sum_{n=1}^{\infty} |\langle \mathbf{x},\ \mathbf{u}_n\rangle| \coloneqq \sum_{n=1}^{\infty} |a_n|.$$

$\implies$: Given a vector $\mathbf{x} = \sum_n a_n \mathbf{u}_n$, we can immediately note that both $\|\mathbf{x}\|_H < \infty$ and $\langle \mathbf{x},\ \mathbf{u}_n\rangle = a_n$. Since $\{\mathbf{u}_n\}$ being a basis is equivalent to Parseval's identity holding, we immediately obtain

$$\sum_{n=1}^{\infty} |a_n| = \sum_{n=1}^{\infty} |\langle \mathbf{x},\ \mathbf{u}_n\rangle| = \|\mathbf{x}\|_H^2 < \infty.$$

5.2 Part b

In both cases, suppose such a linear functional exists.

1. Using part (a), we know that H is isometrically isomorphic to $\ell^2(\mathbb{N})$, and thus $H_f^\vee \cong (\ell^2(\mathbb{N}))^\vee \cong_d \ell^2(\mathbb{N})$.

Note: this follows since $\ell^p(\mathbb{N})^\vee \cong \ell^q(\mathbb{N})$ where p, q are Holder conjugates.

But then, since $L \in H^\vee$, under the isometry f it maps to the functional

$$\begin{aligned} L_\ell : \ell^2(\mathbb{Z}) &\to \mathbb{C} \\ \mathbf{a} = \{a_n\} &\mapsto \sum_{n\in\mathbb{N}} a_n n^{-1}, \end{aligned}$$

which under the identification of dual spaces g identifies L_ℓ with the vector $\mathbf{b} \coloneqq \{n^{-1}\}_{n\in\mathbb{N}}$.

Most importantly, these are all isometries, so we have the equalities

$$\|L\|_H = \|L_\ell\|_{\ell^2(\mathbb{N})^\vee} = \|\mathbf{b}\|_{\ell^2(\mathbb{N})},$$

so it suffices to compute the ℓ^2 norm of the sequence $b_n = \frac{1}{n}$. To this end, we have

$$\begin{aligned} \|\mathbf{b}\|_{\ell^2(\mathbb{N})}^2 &= \sum_n \left|\frac{1}{n}\right|^2 \\ &= \sum_n \frac{1}{n^2} \\ &= \frac{\pi^2}{6}, \end{aligned}$$

which shows that $\|L\|_H = \pi/\sqrt{6}$.

2. Using the same argument, we obtain $\mathbf{b} = \left\{n^{-1/2}\right\}_{n\in\mathbb{N}}$, and thus

$$\|L\|_H^2 = \|\mathbf{b}\|_{\ell^2(\mathbb{N})}^2 = \sum_n \left|n^{-1/2}\right|^2 \to \infty.$$

which shows that L is unbounded, and thus can not be a continuous linear functional. $\square$

6 Problem 6

We can use the fact that $\Lambda_p \in (L^p)^\vee \cong L^q$, where this is an isometric isomorphism given by the map

$$\begin{aligned} I : L^q &\to (L^p)^\vee \\ g &\mapsto (f \mapsto \int fg). \end{aligned}$$

Under this identification, for any $\Lambda \in (L^p)^\vee$, to any $\Lambda \in (L^p)^\vee$ we can associate a $g \in L^q$, where we have

$$\|\Lambda\|_{(L^p)^\vee} = \|g\|_{L^q}.$$

In this case, we can identify $\Lambda_p = I(g)$, where $g(x) = x^2$ and we can verify that $g \in L^q$ by computing its norm:

$$\begin{aligned} \|g\|_{L^q}^q &= \int_0^1 (x^2)^q \; dx \\ &= \left.\frac{x^{2q+1}}{2q+1}\right|_0^1 \\ &= \frac{1}{2q+1} \\ &= \frac{p-1}{3p-1} < \infty, \end{aligned}$$

where we identify $q = \frac{p}{p-1}$, and note that this is finite for all $1 \leq p \leq \infty$ since it limits to $\frac{1}{3}$.

But then

$$\|\Lambda_p\|_{(L^p)^\vee} = \|g\|_{L^q} = \left(\frac{p-1}{3p-1}\right)^{\frac{1}{q}} = \left(\frac{p-1}{3p-1}\right)^{\frac{p-1}{p}},$$

which shows that Λ_p is bounded and thus a continuous linear functional. $\square$