Math 8100 Assignment 1
Preliminaries

Due date: Tuesday the 27th of August 2019

1. The **Cantor set** $\mathcal{C}$ is the set of all $x \in [0,1]$ that have a ternary expansion $x = \sum_{k=1}^{\infty} a_k 3^{-k}$ with $a_k \neq 1$ for all k. Thus $\mathcal{C}$ is obtained from $[0,1]$ by removing the open middle third $(\frac{1}{3}, \frac{2}{3})$, then removing the open middle thirds $(\frac{1}{9}, \frac{2}{9})$ and $(\frac{7}{9}, \frac{8}{9})$ of the two remaining intervals, and so forth.

   (a) Find a real number x belonging to the Cantor set which is not the endpoint of one of the intervals used in its construction.

   (b) Prove that $\mathcal{C}$ is both nowhere dense (and hence meager) and has measure zero.

   (c) Prove that $\mathcal{C}$ is uncountable by showing that the function $f(x) = \sum_{k=1}^{\infty} b_k 2^{-k}$ where $b_k = a_k/2$, maps $\mathcal{C}$ onto $[0,1]$.

2. A set $A \subseteq \mathbb{R}^n$ is called an F_σ set if it can be written as the countable union of closed subsets of $\mathbb{R}^n$. A set $B \subseteq \mathbb{R}^n$ is called a G_δ set if it can be written as the countable intersection of open subsets of $\mathbb{R}^n$.

   (a) Argue that a set is a G_δ set if and only if its complement is an F_σ set.

   (b) Show that every closed set is a G_δ set and every open set is an F_σ set.
   Hint: One approach is to prove that every open subset of $\mathbb{R}^n$ can be written as a countable union of closed cubes with disjoint interiors. This approach is however very specific to open sets in $\mathbb{R}^n$.

   (c) Give an example of an F_σ set which is not a G_δ set and a set which is neither an F_σ nor a G_δ set.

3. (a) Let $\{r_n\}_{n=1}^{\infty}$ be any enumeration of all the rationals in $[0,1]$ and define $f : [0,1] \to \mathbb{R}$ by setting

$$f(x) = \begin{cases} \frac{1}{n} & \text{if } x = r_n \\ 0 & \text{if } x \in [0,1] \setminus \mathbb{Q} \end{cases}.$$

   Prove that $\lim_{x \to c} f(x) = 0$ for every $c \in [0,1]$ and conclude that set of all points at which f is discontinuous is precisely $[0,1] \cap \mathbb{Q}$.

   (b) Let $f : \mathbb{R} \to \mathbb{R}$ be bounded.

   i. Recall that we defined the *oscillation of f at x* to be

$$\omega_f(x) := \lim_{\delta \to 0^+} \sup_{y,z \in B_\delta(x)} |f(y) - f(z)|.$$

   Briefly explain why this is a well defined notion and prove that

$$f \text{ is continuous at } x \iff \omega_f(x) = 0.$$

   ii. Prove that for every $\varepsilon > 0$ the set $A_\varepsilon = \{x \in \mathbb{R} : \omega_f(x) \geq \varepsilon\}$ is closed and deduce from this that the set of all points at which f is discontinuous is an F_σ set.

4. Let $\{x_n\}_{n=1}^{\infty}$ be any enumeration of a given countable set $X \subseteq \mathbb{R}$. For each $n \in \mathbb{N}$ define

$$f_n(x) = \begin{cases} 1 \text{ if } x > x_n \\ 0 \text{ if } x \leq x_n \end{cases}.$$

   Prove that

$$f(x) = \sum_{n=1}^{\infty} \frac{1}{n^2} f_n(x)$$

   defines an increasing function f on $\mathbb{R}$ that is continuous on $\mathbb{R} \setminus X$.

5. Let $C([0,1])$ denote the collection of all real-valued continuous functions with domain $[0,1]$.
   (a) Show that $d_\infty(f,g) = \sup_{x\in[0,1]} |f(x)-g(x)|$ defines a metric on $C([0,1])$ and that with the "uniform" metric $C([0,1])$ is in fact a *complete* metric space.
   (b) Prove that the unit ball $\{f \in C([0,1]) : d_\infty(f,0) \le 1\}$ is closed and bounded, but *not* compact.
   (c) ** Challenge: Can you show that $C([0,1])$ with the metric d_∞ is not *totally bounded*.
   A set is totally bounded if, for every $\varepsilon > 0$, it can be covered by finitely many balls of radius ε.

6. Let
$$g(x) = \sum_{n=0}^{\infty} \frac{1}{1+n^2 x}.$$
   (a) Show that the series defining g does not converge uniformly on $(0,\infty)$, but none the less still defines a continuous function on $(0,\infty)$.
   Hint for the first part: Show that if $\sum_{n=0}^{\infty} g_n(x)$ converges uniformly on a set X, then the sequence of functions $\{g_n\}$ must converge uniformly to 0 on X.
   (b) Is g differentiable on $(0,\infty)$? If so, is the derivative function g' continuous on $(0,\infty)$?

7. Let $h_n(x) = \dfrac{x}{(1+x)^{n+1}}$.
   (a) Prove that h_n converges uniformly to 0 on $[0,\infty)$.
   (b) i. Verify that
$$\sum_{n=0}^{\infty} h_n(x) = \begin{cases} 1 \text{ if } x > 0 \\ 0 \text{ if } x = 0 \end{cases}$$
   ii. Does $\sum_{n=0}^{\infty} h_n$ converge uniformly on $[0,\infty)$?
   (c) Prove that $\sum_{n=0}^{\infty} h_n$ converges uniformly on $[a,\infty)$ for any $a > 0$.

Extra Challenge Problems

Not to be handed in with the assignment

1. Given an arbitrary F_σ set V, can you produce a function whose discontinuities lie precisely in V?
   Hint: First try to do this for an arbitrary closed set.

2. (Baire Category Theorem) Prove that if X is a non-empty *complete* metric space, then X cannot be written as a countable union of nowhere dense sets.
   Hint: Modify the proof given in class of the special case $X = \mathbb{R}$ replacing the use of the nested interval property with the following fact (which you should prove):
   If $F_1 \supseteq F_2 \supseteq \cdots$ is a nested sequence of closed non-empty and bounded sets in a complete metric space X with $\lim_{n\to\infty} \operatorname{diam} F_n = 0$, then $\bigcap_{n=1}^{\infty} F_n$ contains exactly one point.

3. Complete the proof, sketched in class, of the so-called Lebesgue Criterion: *A bounded function on an interval $[a,b]$ is Riemann integrable if and only if its set of discontinuities has measure zero.*
   (a) Prove that if the set of discontinuities of f has measure zero, then f is Riemann integrable.
   [*Hint: Let $\varepsilon > 0$. Cover the compact set A_ε (defined in Q3(b)ii. above) by a finite number of open intervals whose total length is $\le \varepsilon$. Select and appropriate partition of $[a,b]$ and estimate the difference between the upper and lower sums of f over this partition.*]
   (b) Prove that if f is Riemann integrable on $[a,b]$, then its set of discontinuities has measure zero.
   [*Hint: The set of discontinuities of f is contained in $\bigcup_n A_{1/n}$. Given $\varepsilon > 0$, choose a partition P such that $U(f,P) - L(f,P) < \varepsilon/n$. Show that the total length of the intervals in P whose interiors intersect $A_{1/n}$ is $\le \varepsilon$.*]

Math 8100 Assignment 2
Lebesgue measure and outer measure

Due date: Wednesday the 5th of September 2018

1. Prove that if $E \subseteq \mathbb{R}$ with $m_*(E) = 0$, then $E^2 := \{x^2 \mid x \in E\}$ also has Lebesgue outer measure zero.
   Hint: First consider the case when E is a bounded subset of $\mathbb{R}$.
   [*To what extent can you generalize this result?*]

2. Prove that if E_1 and E_2 are measurable subsets of $\mathbb{R}^n$, then
$$m(E_1 \cup E_2) + m(E_1 \cap E_2) = m(E_1) + m(E_2).$$

3. Suppose that $A \subseteq E \subseteq B$, where A and B are Lebesgue measurable subsets on $\mathbb{R}^n$.
   (a) Prove that if $m(A) = m(B) < \infty$, then E is measurable.
   (b) Give an example showing that the same conclusion does not hold if A and B have infinite measure.

4. Suppose A and B are a pair of compact subsets of $\mathbb{R}^n$ with $A \subseteq B$, and let $a = m(A)$ and $b = m(B)$. Prove that for any c with $a < c < b$, there is a compact set E with $A \subseteq E \subseteq B$ and $m(E) = c$.
   Hint: As a warm-up example, consider the one dimensional example where A a compact measurable subset of $B := [0, 1]$ and the quantity $m(A) + t - m(A \cap [0, t])$ as a function of t.

5. Let $\mathcal{N}$ denote the non-measurable subset of $[0, 1]$ that was constructed in lecture.
   (a) Prove that if E is a measurable subset of $\mathcal{N}$, then $m(E) = 0$.
   (b) Show that $m_*([0, 1] \setminus \mathcal{N}) = 1$
   [*Hint: Argue by contradiction and pick an open set G such that $[0, 1] \setminus \mathcal{N} \subseteq G \subseteq [0, 1]$ with $m_*(G) \leq 1 - \varepsilon$.*]
   (c) Conclude that there exists *disjoint* sets $E_1 \subseteq [0, 1]$ and $E_2 \subseteq [0, 1]$ for which
$$m_*(E_1 \cup E_2) \neq m_*(E_1) + m_*(E_2).$$

6. (a) **The Borel-Cantelli Lemma.** Suppose $\{E_j\}_{j=1}^{\infty}$ is a countable family of measurable subsets of $\mathbb{R}^n$ and that
$$\sum_{j=1}^{\infty} m(E_j) < \infty.$$
   Let
$$E = \limsup_{j\to\infty} E_j := \{x \in \mathbb{R}^n \,:\, x \in E_j, \text{ for infinitely many } j\}.$$
   Show that E is measurable and that $m(E) = 0$. *Hint: Write $E = \cap_{k=1}^{\infty} \cup_{j\geq k} E_j$.*
   (b) Given any irrational x one can show (using the pigeonhole principle, for example) that there exists infinitely many fractions a/q, with a and q relatively prime integers, such that
$$\left|x - \frac{a}{q}\right| \leq \frac{1}{q^2}.$$
   However, show that the set of those $x \in \mathbb{R}$ such that there exists infinitely many fractions a/q, with a and q relatively prime integers, such that
$$\left|x - \frac{a}{q}\right| \leq \frac{1}{q^3}$$
   is a set of Lebesgue measure zero.

Extra Challenge Problems

Not to be handed in with the assignment

1. Prove that any $E \subset \mathbb{R}$ with $m_*(E) > 0$ necessarily contains a non-measurable set.

2. The **outer Jordan content** $J_*(E)$ of a set E in $\mathbb{R}$ is defined by

$$J_*(E) = \inf \sum_{j=1}^{N} |I_j|,$$

where the infimum is taken over every *finite* covering $E \subseteq \cup_{j=1}^{N} I_j$, by intervals I_j.

   (a) Prove that $J_*(E) = J_*(\bar{E})$ for every set E (here $\bar{E}$ denotes the closure of E).

   (b) Exhibit a countable subset $E \subseteq [0,1]$ such that $J_*(E) = 1$ while $m_*(E) = 0$.

3. If I is a bounded interval and $\alpha \in (0,1)$, let us call the open interval with the same midpoint as I and length equal to α times the length of I the "open middle αth" of I. If $\{\alpha_j\}_{j=1}^{\infty}$ is any sequence of numbers in $(0,1)$, then, we can define a decreasing sequence $\{K_j\}$ of closed sets as follows: $K_0 = [0,1]$, and K_j is obtained by removing the the open middle α_jth from each of the intervals that make up K_{j-1}. The resulting limiting set $K = \bigcap_{j=1}^{\infty} K_j$ is called a **generalized Cantor set**.

   (a) Suppose $\{\alpha_j\}_{j=1}^{\infty}$ is any sequence of numbers in $(0,1)$.

      i. Prove that $\prod_{j=1}^{\infty}(1-\alpha_j) > 0$ if and only if $\sum_{j=1}^{\infty} \alpha_j < \infty$.

      ii. Given $\beta \in (0,1)$, exhibit a sequence $\{\alpha_j\}$ such that $\prod_{j=1}^{\infty}(1-\alpha_j) = \beta$.

   (b) Given $\beta \in (0,1)$, construct an open set G in $[0,1]$ whose boundary has Lebesgue measure β.

   Hint: Every closed nowhere dense set is the boundary of an open set.

Math 8100 Assignment 3
Lebesgue measurable sets and functions

Due date: 5:00 pm Friday the 20th of September 2019

1. (a) Prove that for every $E \subseteq \mathbb{R}^n$ there exists a Borel set $B \supseteq E$ with the property that $m(B) = m_*(E)$.
   (b) Prove that if $E \subseteq \mathbb{R}^n$ is Lebesgue measurable, then there exists a Borel set $B \subseteq E$ with the property that $m(B) = m(E)$.
   (c) Prove that if $E \subseteq \mathbb{R}^n$ is Lebesgue measurable with $m(E) < \infty$, then for every $\varepsilon > 0$ there exists a set A that is a finite union of closed cubes such that $m(E \triangle A) < \varepsilon$.
   [*Recall that* $E \triangle A$ *stands for the symmetric difference, defined by* $E \triangle A = (E \setminus A) \cup (A \setminus E)$]

2. Let E be a Lebesgue measurable subset of $\mathbb{R}^n$ with $m(E) > 0$ and $\varepsilon > 0$.
   (a) Prove that E "almost" contains a closed cube in the sense that there exists a closed cube Q such that $m(E \cap Q) \geq (1 - \varepsilon) m(Q)$.
   (b) Prove that the so-called difference set $E - E := \{d : d = x - y \text{ with } x, y \in E\}$ necessarily contains an open ball centered at the origin.
   Hint: It may be useful to observe that $d \in E - E \iff E \cap (E + d) \neq \emptyset$.

3. We say that a function $f : \mathbb{R}^n \to \mathbb{R}$ is *upper semicontinuous* at a point x in $\mathbb{R}^n$ if
$$f(x) \geq \limsup_{y \to x} f(y).$$
   Prove that if f is upper semicontinuous at every point x in $\mathbb{R}^n$, then f is Borel measurable.

4. Let $\{f_n\}$ be a sequence of measurable functions on $\mathbb{R}^n$. Prove that $\{x \in \mathbb{R}^n : \lim_{n\to\infty} f_n(x) \text{ exists}\}$ defines a measurable set.

5. Recall that the **Cantor set** $\mathcal{C}$ is the set of all $x \in [0,1]$ that have a ternary expansion $x = \sum_{k=1}^{\infty} a_k 3^{-k}$ with $a_k \neq 1$ for all k. Consider the function
$$f(x) = \sum_{k=1}^{\infty} b_k 2^{-k} \quad \text{where} \quad b_k = a_k/2.$$
   (a) Show that f is well defined and continuous on $\mathcal{C}$, and moreover $f(0) = 0$ as well as $f(1) = 1$.
   (b) Prove that there exists a continuous function that maps a measurable set to a non-measurable set.

6. Let us examine the map f defined in Question 5 even more closely. One readily sees that if $x, y \in \mathcal{C}$ and $x < y$, then $f(x) < f(y)$ unless x and y are the two endpoints of one of the intervals removed from $[0,1]$ to obtain $\mathcal{C}$. In this case $f(x) = \ell 2^m$ for some integers ℓ and m, and $f(x)$ and $f(y)$ are the two binary expansions of this number. We can therefore extend f to a map $F : [0,1] \to [0,1]$ by declaring it to be constant on each interval missing from $\mathcal{C}$. F is called the **Cantor-Lebesgue function**.
   (a) Prove that F is non-decreasing and continuous.
   (b) Let $G(x) = F(x) + x$. Show that G is a bijection from $[0,1]$ to $[0,2]$.
   (c) i. Show that $m(G(\mathcal{C})) = 1$.
      ii. By considering rational translates of $\mathcal{N}$ (the non-measurable subset of $[0,1]$ that we constructed in class), prove that $G(\mathcal{C})$ necessarily contains a (Lebesgue) non-measurable set $\mathcal{N}'$.
      iii. Let $E = G^{-1}(\mathcal{N}')$. Show that E is Lebesgue measurable, but not Borel.
   (d) Give an example of a measurable function φ such that $\varphi \circ G^{-1}$ is not measurable.
   Hint: Let φ *be the characteristic function of a null set whose image under* G *is not measurable.*

Extra Challenge Problems

Not to be handed in with the assignment

1. Let $\chi_{[0,1]}$ be the characteristic function of $[0,1]$. Show that there is no function f satisfying $f = \chi_{[0,1]}$ almost everywhere which is also continuous on all of $\mathbb{R}$.

2. Question 6d above supplies us with an example that if f and g are Lebesgue measurable, then it does not necessarily follow that $f \circ g$ will be Lebesgue measurable, even if g is assumed to be continuous.

   Prove that if f is Borel measurable, then $f \circ g$ will be Lebesgue or Borel measurable whenever g is.

3. Let f be a measurable function on $[0,1]$ with $|f(x)| < \infty$ for a.e. x. Prove that there exists a sequence of continuous functions $\{g_n\}$ on $[0,1]$ such that $g_n \to f$ for a.e. $x \in [0,1]$.

Math 8100 Assignment 4
Lebesgue Integration

Due date: Tuesday the 1st of October 2019

Definition. Let E be a Lebesgue measurable subset of $\mathbb{R}^n$.

We say that a measurable function $f : E \to \mathbb{C}$ is *integrable on E* if $\displaystyle\int_E |f(x)|\,dx < \infty$.

1. (a) Give an example of a continuous integrable function f on $\mathbb{R}$ for which $f(x) \nrightarrow 0$ as $|x| \to \infty$.

   (b) Prove that if f is integrable on $\mathbb{R}$ and uniformly continuous, then $\displaystyle\lim_{|x|\to\infty} f(x) = 0$.

2. Let f be an integrable function on $\mathbb{R}^n$.

   (a) Prove that $\{x : |f(x)| = \infty\}$ has measure equal to zero.

   (b) Let $\varepsilon > 0$. Prove that there exists a measurable set E with $m(E) < \infty$ for which

$$\int_E |f| > \left(\int |f|\right) - \varepsilon.$$

3. Let f be a function in $L^+(\mathbb{R}^n)$ that is finite almost everywhere.

   Let $E_{2^k} = \{x : f(x) > 2^k\}$, $F_k = \{x : 2^k < f(x) \le 2^{k+1}\}$, and note that since f is finite almost everywhere it follows that $\bigcup_{k=-\infty}^{\infty} F_k = \{x : f(x) > 0\}$, and the sets F_k are disjoint. Prove that

$$\int f(x) < \infty \iff \sum_{k=-\infty}^{\infty} 2^k m(F_k) < \infty \iff \sum_{k=-\infty}^{\infty} 2^k m(E_{2^k}) < \infty.$$

4. Prove the following:

   (a)

$$\int_{\{x\in\mathbb{R}^n\,:\,|x|\le 1\}} |x|^{-p}\,dx < \infty \quad \text{if and only if} \quad p < n.$$

   (b)

$$\int_{\{x\in\mathbb{R}^n\,:\,|x|\ge 1\}} |x|^{-p}\,dx < \infty \quad \text{if and only if} \quad p > n.$$

   Hint: One possible approach is to use the first equivalence in Question 3 above. I suggest however that in this case you also try simply writing $\mathbb{R}^n$ as a disjoint union of the annuli $A_k = \{2^k < |x| \le 2^{k+1}\}$.

5. Given any integrable function f on $\mathbb{R}^n$, the *Fourier transform of f* is defined by

$$\widehat{f}(\xi) = \int_{\mathbb{R}^n} f(x) e^{-2\pi i x\cdot\xi}\,dx$$

   where $x \cdot \xi = x_1\xi_1 + \cdots + x_n\xi_n$. Show that $\widehat{f}$ is a bounded continuous function of ξ.

6. Let $\{f_k\}$ be a sequence of integrable functions on $\mathbb{R}^n$, f be integrable on $\mathbb{R}^n$, and $\displaystyle\lim_{k\to\infty} f_k = f$ a.e.

   (a) Suppose further that

$$\lim_{k\to\infty} \int |f_k(x)|\,dx = A < \infty \qquad \text{and} \qquad \int |f(x)|\,dx = B.$$

i. Prove that

$$\lim_{k\to\infty} \int |f_k(x) - f(x)|\, dx = A - B.$$

Hint: Use the fact that

$$|f_k(x)| - |f(x)| \le |f_k(x) - f(x)| \le |f_k(x)| + |f(x)|.$$

ii. Give an example of a sequence $\{f_k\}$ of such functions for which $A \neq B$.

(b) Deduce that

$$\int |f - f_k| \to 0 \quad \Longleftrightarrow \quad \int |f_k| \to \int |f|.$$

7. (a) Suppose that $f(x)$ and $xf(x)$ are both integrable functions on $\mathbb{R}$. Prove that the function

$$F(t) = \int_{\mathbb{R}} f(x) \cos(tx)\, dx.$$

is differentiable at every t and find a formula for $F'(t)$.

(b) Giving complete justification, evaluate

$$\lim_{t\to 0} \int_0^1 \frac{e^{t\sqrt{x}} - 1}{t}\, dx.$$

Extra Challenge Problems

Not to be handed in with the assignment

1. Assume Fatou's theorem and deduce the monotone convergence theorem from it.

2. A sequence $\{f_k\}$ of integrable functions on $\mathbb{R}^n$ is said to *converge in measure* to f if for every $\varepsilon > 0$,

$$\lim_{k\to\infty} m(\{x \in \mathbb{R}^n \,:\, |f_k(x) - f(x)| \ge \varepsilon\}) = 0.$$

(a) Prove that if $f_k \to f$ in L^1 then $f_k \to f$ in measure.

(b) Give an example to show that the converse of Question 2a is false.

(c) Prove that if we make the additional assumption that there exists an integrable function g such that $|f_k| \le g$ for all k, then $f_k \to f$ in measure implies that

i. * (Bonus points) $f \in L^1$
Hint: First show that $\{f_k\}$ contains a subsequence which converges to f almost everywhere.

ii. $f_k \to f$ in L^1.
Hint: Try using absolute continuity and "small tails property" of the Lebesgue integral.

3. Let $\Omega \subseteq \mathbb{R}^n$ be measurable with $m(\Omega) < \infty$. A set $\Phi \subseteq L^1(\Omega)$ is said to be *uniformly integrable* if, for any $\varepsilon > 0$ there exists $\delta > 0$ such that whenever $f \in \Phi$ and $E \subseteq \Omega$ is measurable with $m(E) < \delta$, then

$$\int_E |f(x)|\, dx < \varepsilon.$$

(a) Prove that if $f \in L^1(\Omega)$ and $\{f_k\}$ is a uniformly integrable sequence of functions in $L^1(\Omega)$ such that $f_k \to f$ almost everywhere on Ω, then $f_k \to f$ in $L^1(\Omega)$.

(b) Is it necessary to assume that $f \in L^1(\Omega)$?

Math 8100 Assignment 5
Repeated Integration

Due date: Friday the 18th of October 2019

1. Prove that if $\{a_{jk}\}_{(j,k)\in\mathbb{N}\times\mathbb{N}}$ is a "double sequence" with $a_{jk}\geq 0$ for all $(j,k)\in\mathbb{N}\times\mathbb{N}$, then

$$\sum_{j=1}^{\infty}\sum_{k=1}^{\infty}a_{jk}=\sup\Big\{\sum_{(j,k)\in B}a_{jk}\,:\,B\text{ is a finite subset of }\mathbb{N}\times\mathbb{N}\Big\}$$

and deduce from this that

$$\sum_{j=1}^{\infty}\sum_{k=1}^{\infty}a_{jk}=\sum_{k=1}^{\infty}\sum_{j=1}^{\infty}a_{jk}.$$

This conclusion holds more generally provided $\sum_{j=1}^{\infty}\sum_{k=1}^{\infty}|a_{jk}|<\infty$*, see Theorem 8.3 in "Baby Rudin".*

2. Let $f\in L^1([0,1])$, and for each $x\in[0,1]$ define

$$g(x)=\int_x^1\frac{f(t)}{t}\,dt.$$

Show that $g\in L^1([0,1])$ and that

$$\int_0^1 g(x)\,dx=\int_0^1 f(x)\,dx.$$

3. Carefully prove that if we define

$$f(x,y):=\begin{cases}\dfrac{x^{1/3}}{(1+xy)^{3/2}} & \text{if } 0\leq x\leq y\\ 0 & \text{otherwise}\end{cases}$$

for each $(x,y)\in\mathbb{R}^2$, then f defines a function in $L^1(\mathbb{R}^2)$.

4. Let $A,B\subseteq\mathbb{R}^n$ be bounded measurable sets with positive Lebesgue measure. For each $t\in\mathbb{R}^n$ define the function

$$g(t)=m\left(A\cap(t-B)\right)$$

where $t-B=\{t-b\,:\,b\in B\}$.

(a) Prove that g is a continuous function and

$$\int_{\mathbb{R}^n}g(t)\,dt=m(A)\,m(B).$$

(b) Conclude that the sumset

$$A+B=\{a+b\,:\,a\in A\text{ and }b\in B\}$$

contains a non-empty open subset of $\mathbb{R}^n$.

5. Let $f,g\in L^1([0,1])$ and for each $0\leq x\leq 1$ define

$$F(x):=\int_0^x f(y)\,dy\quad\text{and}\quad G(x):=\int_0^x g(y)\,dy.$$

Prove that

$$\int_0^1 F(x)g(x)\,dx=F(1)G(1)-\int_0^1 f(x)G(x)\,dx.$$

6. Let $f \in L^1(\mathbb{R})$. For any $h > 0$ we define

$$A_h(f)(x) := \frac{1}{2h} \int_{x-h}^{x+h} f(y)\, dy$$

(a) Prove that for all $h > 0$,

$$\int_{\mathbb{R}} |A_h(f)(x)|\, dx \leq \int_{\mathbb{R}} |f(x)|\, dx.$$

(b) Prove that

$$\lim_{h \to 0^+} \int_{\mathbb{R}} |A_h(f)(x) - f(x)|\, dx = 0.$$

One can in fact show that $\lim_{h\to 0^+} A_h(f) = f$ *almost everywhere. This result is actually equivalent to the Lebesgue Density Theorem in* $\mathbb{R}$ *and we will establish this later in the course.*

Extra Challenge Problems

Not to be handed in with the assignment

1. (a) Prove that

$$\int_0^\infty \left| \frac{\sin x}{x} \right| dx = \infty.$$

(b) By considering the iterated integral

$$\int_0^\infty \left(\int_0^\infty x e^{-xy}(1 - \cos y)\, dy \right) dx$$

show (with justification) that

$$\lim_{A \to \infty} \int_0^A \frac{\sin x}{x}\, dx = \frac{\pi}{2}.$$

2. Suppose that F is a closed subset of $\mathbb{R}$ whose complement has finite measure. Let $\delta(x)$ denote the distance from x to F, namely

$$\delta(x) = d(x, F) = \inf\{|x - y| \,:\, y \in F\}$$

and

$$I_F(x) = \int_{-\infty}^{\infty} \frac{\delta(y)}{|x - y|^2}\, dy.$$

(a) Prove that δ is continuous, by showing that it satisfies the Lipschitz condition $|\delta(x) - \delta(y)| \leq |x - y|$.

(b) Show that $I_F(x) = \infty$ if $x \notin F$.

(c) Show that $I_F(x) < \infty$ for a.e. $x \in F$, by showing that $\int_F I_F(x)\, dx < \infty$.

Math 8100 Assignment 6
The Fourier Transform

Due date: Thursday the 31st of October 2019

Recall that we have defined the Fourier transform of an integrable function f on $\mathbb{R}^n$ by

$$\widehat{f}(\xi) = \int_{\mathbb{R}^n} f(x)e^{-2\pi i x\cdot\xi}\, dx$$

where $x \cdot \xi = x_1\xi_1 + \cdots + x_n\xi_n$ and the convolution of two integrable functions f and g on $\mathbb{R}^n$ by

$$f * g(x) = \int_{\mathbb{R}^n} f(x-y)g(y)\, dy.$$

1. Prove that if $f \in L^1(\mathbb{R}^n)$, then $\widehat{f}(\xi) \to 0$ as $|\xi| \to \infty$. (This is called the Riemann-Lebesgue lemma.)
   Hint: Write $\widehat{f}(\xi) = \frac{1}{2}\int[f(x) - f(x-\xi')]e^{-2\pi i x\cdot\xi}\, dx$, where $\xi' = \frac{\xi}{2|\xi|^2}$.

2. (a) Prove that if $f, g \in L^1(\mathbb{R}^n)$, then $\widehat{f * g}(\xi) = \widehat{f}(\xi)\widehat{g}(\xi)$ for all $\xi \in \mathbb{R}^n$.
   (b) Conclude from part (a) that
      i. if $f, g, h \in L^1(\mathbb{R}^n)$, then $f * g = g * f$ and $(f * g) * h = f * (g * h)$ almost everywhere.
      ii. there does not exist $I \in L^1(\mathbb{R}^n)$ such that $f * I = f$ almost everywhere for all $f \in L^1(\mathbb{R}^n)$.

3. Let $f \in L^1(\mathbb{R}^n)$.
   (a) Show that if $y \in \mathbb{R}^n$ and
      i. $g(x) = f(x-y)$ for all $x \in \mathbb{R}^n$, then $\widehat{g}(\xi) = e^{-2\pi i y\cdot\xi}\widehat{f}(\xi)$ for all $\xi \in \mathbb{R}^n$.
      ii. $h(x) = e^{2\pi i x\cdot y}f(x)$ for all $x \in \mathbb{R}^n$, then $\widehat{h}(\xi) = \widehat{f}(\xi - y)$ for all $\xi \in \mathbb{R}^n$.
   (b) Show that if T be a non-singular linear transformation of $\mathbb{R}^n$ and $S = (T^*)^{-1}$ denote its inverse transpose, then
   $$\widehat{f \circ T}(\xi) = \frac{1}{|\det T|}\widehat{f}(S\xi)$$
   for all $\xi \in \mathbb{R}^n$.

4. (a) Let $f \in L^1(\mathbb{R})$.
      i. Let $g(x) = xf(x)$. Show that if $g \in L^1$, then $\widehat{f}$ is differentiable and $\frac{d}{d\xi}\widehat{f}(\xi) = -2\pi i\,\widehat{g}(\xi)$.
      ii. Let $f \in C_0^1(\mathbb{R})$ and $h(x) = \frac{d}{dx}f(x)$. Show that if $h \in L^1$, then $\widehat{h}(\xi) = 2\pi i\xi\widehat{f}(\xi)$.
      Recall that $C_0^1(\mathbb{R})$ is the collection of functions in $C^1(\mathbb{R})$ which vanishes at infinity.
   (b) Let $G(x) = e^{-\pi x^2}$. By considering the derivative of $\widehat{G}(\xi)/G(\xi)$, show that $\widehat{G}(\xi) = G(\xi)$.
   Hint: You may also want to use the fact that $\int_{\mathbb{R}} G(x)\, dx = 1$ (see "challenge" problem).

5. The functions D, F, and P defined below are all bounded $L^+(\mathbb{R})$ functions with integrals equal to 1.
   (a) Show that if
   $$D(x) = \begin{cases} 1 & \text{if } |x| \le 1/2 \\ 0 & \text{otherwise} \end{cases}$$
   then
   $$\widehat{D}(\xi) = \frac{\sin \pi\xi}{\pi\xi}.$$
   This gives, in light of Assignment 5 Challenge Problem 1(a), an explicit example of a function which is not in $L^1(\mathbb{R})$, but yet is the Fourier transform of an L^1 function. See Question 6 for additional higher dimensional examples.

(b) Let

$$F(x) = \begin{cases} 1 - |x| & \text{if } |x| \leq 1 \\ 0 & \text{otherwise} \end{cases}.$$

i. Show that

$$\widehat{F}(\xi) = \left(\frac{\sin \pi\xi}{\pi\xi}\right)^2.$$

Hint: It may help to write $\widehat{F}(\xi) = h(\xi) + h(-\xi)$ *where* $h(\xi) = e^{2\pi i\xi} \int_0^1 ye^{-2\pi iy\xi} dy$.

ii. Find the Fourier transform of the function

$$f(x) = \left(\frac{\sin \pi x}{\pi x}\right)^2.$$

Be careful to fully justify your answer.

(c) Show that if

$$P(x) = \frac{1}{\pi}\frac{1}{1+x^2}.$$

then

$$\int_{-\infty}^{\infty} e^{-2\pi|\xi|} e^{2\pi ix\xi}\, d\xi = P(x)$$

and hence that

$$\widehat{P}(\xi) = e^{-2\pi|\xi|}.$$

Be careful to fully justify your answer.

Remark: *In Questions 4b and 5 above D is for Dirichlet, F is for Fejér, P is for Poisson, and G is for Gauss-Weierstrass. The respective "approximate identities", namely* $\{(\widehat{D})_t\}_{t>0}$, $\{(\widehat{F})_t\}_{t>0}$, $\{P_t\}_{t>0}$, *and* $\{G_{\sqrt{t}}\}_{t>0}$, *are generally referred to as Dirichlet, Fejér, Poisson, and Gauss-Weierstrass kernels.*

6. Show that for any $\varepsilon > 0$ the function $F(\xi) = (1+|\xi|^2)^{-\varepsilon}$ is the Fourier transform of an $L^1(\mathbb{R}^n)$ function.
Hint: Consider the function

$$f(x) = \int_0^\infty G_t(x) e^{-\pi t^2} t^{2\varepsilon-1}\, dt,$$

where $G_t(x) = t^{-n}e^{-\pi|x|^2/t^2}$. *Now use Fubini/Tonelli to prove that* $f \in L^1(\mathbb{R}^n)$ *with* $\widehat{f}(\xi) = F(\xi)\|f\|_1$.

Extra Challenge Problems

Not to be handed in with the assignment

1. By considering the iterated integral

$$\int_0^\infty \left(\int_0^\infty xe^{-x^2(1+y^2)}\, dx\right) dy$$

show (with justification) that

$$\int_0^\infty e^{-x^2}\, dx = \frac{\sqrt{\pi}}{2}$$

and hence that

$$\int_{-\infty}^\infty e^{-\pi x^2}\, dx = 1.$$

Math 8100 Assignment 7
Hilbert Spaces

Due date: Thursday 14th of November 2019

1. (a) Prove that $\ell^2(\mathbb{N})$ is complete.

   Recall that $\ell^2(\mathbb{N}) := \{x = \{x_j\}_{j=1}^{\infty} \,:\, \|x\|_{\ell^2} < \infty\}$, *where* $\|x\|_{\ell^2} := \left(\sum_{j=1}^{\infty} |x_j|^2\right)^{1/2}$.

   (b) Let H be a Hilbert space. Prove the so-called *polarization identity*, namely that for any $x, y \in H$,

$$\langle x, y\rangle = \frac{1}{4}\left(\|x+y\|^2 - \|x-y\|^2 + i\|x+iy\|^2 - i\|x-iy\|^2\right)$$

   and conclude that any invertible linear map from H to $\ell^2(\mathbb{N})$ is *unitary* if and only if it is *isometric*.

   Recall that if H_1 and H_2 are Hilbert spaces with inner products $\langle\cdot,\cdot\rangle_1$ and $\langle\cdot,\cdot\rangle_2$, then a mapping $U : H_1 \to H_2$ is said to be **unitary** *if it is an invertible linear map that preserves inner products, namely $\langle Ux, Uy\rangle_2 = \langle x, y\rangle_1$, and an* **isometry** *if it preserves "lengths", namely $\|Ux\|_2 = \|x\|_1$.*

2. Let E be a subset of a Hilbert space H.

   (a) Show that $E^{\perp} := \{x \in H \,:\, \langle x, y\rangle = 0 \text{ for all } y \in E\}$ is a closed subspace of H.

   (b) Show that $(E^{\perp})^{\perp}$ is the smallest closed subspace of H that contains E.

3. In $L^2([0,1])$ let $e_0(x) = 1$, $e_1(x) = \sqrt{3}(2x-1)$ for all $x \in (0,1)$.

   (a) Show that e_0, e_1 is an orthonormal system in $L^2(0,1)$.

   (b) Show that the polynomial of degree 1 which is closest with respect to the norm of $L^2(0,1)$ to the function $f(x) = x^2$ is given by $g(x) = x - 1/6$. What is $\|f - g\|_2$?

4. (a) Verify that the following systems are orthogonal in $L^2([0,1])$:

      i. $\{1/\sqrt{2}, \cos(2\pi x), \sin(2\pi x), \ldots, \cos(2\pi kx), \sin(2\pi kx), \ldots\}$

      ii. $\{e^{2\pi ikx}\}_{k=-\infty}^{\infty}$

   (b) Let $f \in L^1([0,1])$.

      i. Show that for any $\epsilon > 0$ we can write $f = g + h$, where $g \in L^2$ and $\|h\|_1 < \epsilon$.

      ii. Use this decomposition of f to prove the so-called *Riemann-Lebesgue lemma*:

$$\lim_{k\to\infty}\int_0^1 f(x)\cos(2\pi kx)\,dx = \lim_{k\to\infty}\int_0^1 f(x)\sin(2\pi kx)\,dx = 0$$

5. (a) The first three Legendre polynomials are

$$P_0(x) = 1, \quad P_1(x) = x, \quad P_2(x) = (3x^2-1)/2.$$

   Show that the orthonormal system in $L^2([-1,1])$ obtained by applying the Gram-Schmidt process to $1, x, x^2$ are scalar multiples of these.

(b) Compute

$$\min_{a,b,c} \int_{-1}^{1} |x^3 - a - bx - cx^2|^2 \, dx$$

(c) Find

$$\max \int_{-1}^{1} x^3 g(x) \, dx$$

where g is subject to the restrictions

$$\int_{-1}^{1} g(x) \, dx = \int_{-1}^{1} x g(x) \, dx = \int_{-1}^{1} x^2 g(x) \, dx = 0 \, ; \quad \int_{-1}^{1} |g(x)|^2 \, dx = 1.$$

6. Let

$$\mathcal{C} = \left\{ f \in L^2([0,1]) \, : \, \int_0^1 f(x) \, dx = 1 \text{ and } \int_0^1 x f(x) \, dx = 2 \right\}$$

(a) Let $g(x) = 18x^2 - 5$. Show that $g \in \mathcal{C}$ and that

$$\mathcal{C} = g + \mathcal{S}^{\perp}$$

where $\mathcal{S}^{\perp}$ denotes the orthogonal complement of $\mathcal{S} = \text{Span}\,(\{1, x\})$.

(b) Find *the* function $f_0 \in \mathcal{C}$ for which

$$\int_0^1 |f_0(x)|^2 dx = \inf_{f \in \mathcal{C}} \int_0^1 |f(x)|^2 dx.$$

Extra Challenge Problems

Not to be handed in with the assignment

1. Prove that every closed convex set K in a Hilbert space has a unique element of minimal norm.

2. **The Mean Ergodic Theorem:** *Let U be a unitary operator on a Hilbert space H.*

   Prove that if $M = \{x \, : \, Ux = x\}$ and $S_N = \frac{1}{N} \sum_{n=0}^{N-1} U^n$, then $\lim_{N \to \infty} \|S_N x - Px\| = 0$ for all $x \in H$, where Px denotes the orthogonal projection of x onto M.

Math 8100 Assignment 8
Basic Function Spaces

Due date: Tuesday the 26th of November 2019

1. Prove the following basic properties of $L^\infty = L^\infty(X)$, where X is a measurable subset of $\mathbb{R}^n$:
   (a) $\|\cdot\|_\infty$ is a norm on L^∞ and when equipped with this norm L^∞ is a Banach space.
   (b) $\|f_n - f\|_\infty \to 0$ iff there exists $E \in \mathbb{R}^n$ such that $m(E^c) = 0$ and $f_n \to f$ uniformly on E.
   (c) Simple functions are dense in L^∞, but continuous functions with compact support are not.

   Recall that if $X \subseteq \mathbb{R}^n$ is measurable and f is a measurable function on X, then we define
   $$\|f\|_\infty = \inf\{a \geq 0 : m(\{x \in X : |f(x)| > a\}) = 0\},$$
   with the convention that $\inf \emptyset = \infty$, *and*
   $$L^\infty = L^\infty(X) = \{f : X \to \mathbb{C} \text{ measuarable} : \|f\|_\infty < \infty\},$$
   with the usual convention that two functions that are equal a.e. define the same element of L^∞. Thus $f \in L^\infty$ if and only if there is a bounded function g such that $f = g$ almost everywhere; we can take $g = f\chi_E$ where $E = \{x : |f(x)| \leq \|f\|_\infty\}$.

2. Let $X \subseteq \mathbb{R}^n$ be measurable.
   (a) i. Prove that if $m(X) < \infty$, then
   $$L^\infty(X) \subset L^2(X) \subset L^1(X) \tag{1}$$
   with strict inclusion in each case, and that for any measurable $f : X \to \mathbb{C}$ one in fact has
   $$\|f\|_{L^1(X)} \leq m(X)^{1/2}\|f\|_{L^2(X)} \leq m(X)\|f\|_{L^\infty(X)}.$$
   ii. Give examples to show that no such result of the form (1) can hold if one drops the assumption that $m(x) < \infty$. Prove, furthermore, that if $L^2(X) \subseteq L^1(X)$, then $m(X) < \infty$.
   (b) Prove that
   $$\underbrace{L^1(X) \cap L^\infty(X) \subset L^2(X)}_{(\star)} \subset L^1(X) + L^\infty(X)$$
   and that in addition to $(\star)$ one in fact has
   $$\|f\|_{L^2(X)} \leq \|f\|^{1/2}_{L^1(X)}\|f\|^{1/2}_{L^\infty(X)}$$
   for any measurable function $f : X \to \mathbb{C}$.

3. Prove that
   $$\ell^1(\mathbb{Z}) \subset \ell^2(\mathbb{Z}) \subset \ell^\infty(\mathbb{Z})$$
   with strict inclusion in each case, and that for any sequence $a = \{a_j\}_{j\in\mathbb{Z}}$ of complex numbers one in fact has
   $$\|a\|_{\ell^\infty(\mathbb{Z})} \leq \|a\|_{\ell^2(\mathbb{Z})} \leq \|a\|_{\ell^1(\mathbb{Z})}.$$
   Recall that for $p = 1, 2, \infty$ we define
   $$\ell^p(\mathbb{Z}) = \{a = \{a_j\}_{j\in\mathbb{Z}} \subseteq \mathbb{C} : \|a\|_{\ell^p(\mathbb{Z})} < \infty\}$$
   where
   $$\|a\|_{\ell^1(\mathbb{Z})} = \sum_{j=-\infty}^{\infty} |a_j|, \quad \|a\|_{\ell^2(\mathbb{Z})} = \Big(\sum_{j=-\infty}^{\infty} |a_j|^2\Big)^{1/2}, \text{ and } \|a\|_{\ell^\infty(\mathbb{Z})} = \sup_j |a_j|.$$

4. Let $C([0,1])$ denote the space of all continuous real-valued functions on $[0,1]$.

   (a) Prove that $C([0,1])$ is complete under the uniform norm $\|f\|_u := \sup_{x\in[0,1]} |f(x)|$.

   (b) Prove that $C([0,1])$ is not complete under the L^1-norm $\|f\|_1 = \int_0^1 |f(x)|\,dx$

5. Let H be a Hilbert space with orthonormal basis $\{u_n\}_{n=1}^{\infty}$.

   (a) Let $\{a_n\}_{n=1}^{\infty}$ be a sequence of complex numbers. Prove that

$$\sum_{n=1}^{\infty} a_n u_n \text{ converges in } H \iff \sum_{n=1}^{\infty} |a_n|^2 < \infty,$$

   and moreover that if $\sum_{n=1}^{\infty} |a_n|^2 < \infty$, then $\Big\|\sum_{n=1}^{\infty} a_n u_n\Big\| = \Big(\sum_{n=1}^{\infty} |a_n|^2\Big)^{1/2}$.

   (b) i. Is there a continuous linear functional L on H such that $L(u_n) = n^{-1}$ for all $n \in \mathbb{N}$? If L exists, find its norm.

   ii. Is there a continuous linear functional L on H such that $L(u_n) = n^{-1/2}$ for all $n \in \mathbb{N}$? If L exists, find its norm.

6. For each $1 \le p \le \infty$, define $\Lambda_p : L^p([0,1]) \to \mathbb{R}$ by

$$\Lambda_p(f) = \int_0^1 x^2 f(x)\,dx.$$

   Explain why Λ_p is a continuous linear functional and compute its norm (in terms of p).

Extra Practice Problems

Not to be handed in with the assignment

1. Let f and g be two non-negative Lebesgue measurable functions on $[0,\infty)$. Suppose that

$$A := \int_0^{\infty} f(y)\, y^{-1/2} dy < \infty \qquad \text{and} \qquad B := \Big(\int_0^{\infty} |g(y)|^2 dy\Big)^{1/2} < \infty$$

   Prove that

$$\int_0^{\infty} \Big(\int_0^x f(y)\,dy\Big) \frac{g(x)}{x}\,dx \le AB$$

2. Let $\{f_k\}$ be any sequence of functions in $L^2([0,1])$ satisfying $\|f_k\|_2 \le 1$ for all $k \in \mathbb{N}$.

   (a) i. Prove that if $f_k \to f$ either a.e. on $[0,1]$ or in $L^1([0,1])$, then $f \in L^2([0,1])$ with $\|f\|_2 \le 1$.

   ii. Do either of the above hypotheses guarantee that $f_k \to f$ in $L^2([0,1])$?

   (b) Prove that if $f_k \to f$ a.e. on $[0,1]$, then this in fact implies that $f_k \to f$ in $L^1([0,1])$.

3. Let $1 \le p \le \infty$. Prove that if $\{f_k\}_{k=1}^{\infty}$ is a sequence of functions in $L^p(\mathbb{R}^n)$ with the property that

$$\sum_{k=1}^{\infty} \|f_k\|_p < \infty,$$

   then $\sum f_k$ converges almost everywhere to an $L^p(\mathbb{R}^n)$ function with

$$\Big\|\sum_{k=1}^{\infty} f_k\Big\|_p \le \sum_{k=1}^{\infty} \|f_k\|_p.$$