Exercises

1. A family of sets $\mathcal{R} \subset \mathcal{P}(X)$ is called a **ring** if it is closed under finite unions and differences (i.e., if $E_1, \ldots, E_n \in \mathcal{R}$, then $\bigcup_1^n E_j \in \mathcal{R}$, and if $E, F \in \mathcal{R}$, then $E \setminus F \in \mathcal{R}$). A ring that is closed under countable unions is called a **σ-ring**.

- **a.** Rings (resp. σ-rings) are closed under finite (resp. countable) intersections.
- **b.** If $\mathcal{R}$ is a ring (resp. σ-ring), then $\mathcal{R}$ is an algebra (resp. σ-algebra) iff $X \in \mathcal{R}$.
- **c.** If $\mathcal{R}$ is a σ-ring, then $\{E \subset X : E \in \mathcal{R} \text{ or } E^c \in \mathcal{R}\}$ is a σ-algebra.
- **d.** If $\mathcal{R}$ is a σ-ring, then $\{E \subset X : E \cap F \in \mathcal{R} \text{ for all } F \in \mathcal{R}\}$ is a σ-algebra.

2. Complete the proof of Proposition 1.2.

3. Let $\mathcal{M}$ be an infinite σ-algebra.

- **a.** $\mathcal{M}$ contains an infinite sequence of disjoint sets.
- **b.** $\text{card}(\mathcal{M}) \geq \mathfrak{c}$.

4. An algebra $\mathcal{A}$ is a σ-algebra iff $\mathcal{A}$ is closed under countable increasing unions (i.e., if $\{E_j\}_1^\infty \subset \mathcal{A}$ and $E_1 \subset E_2 \subset \cdots$, then $\bigcup_1^\infty E_j \in \mathcal{A}$).

5. If $\mathcal{M}$ is the σ-algebra generated by $\mathcal{E}$, then $\mathcal{M}$ is the union of the σ-algebras generated by $\mathcal{F}$ as $\mathcal{F}$ ranges over all countable subsets of $\mathcal{E}$. (Hint: Show that the latter object is a σ-algebra.)

1.9 Theorem. *Suppose that $(X, \mathcal{M}, \mu)$ is a measure space. Let $\mathcal{N} = \{N \in \mathcal{M} : \mu(N) = 0\}$ and $\overline{\mathcal{M}} = \{E \cup F : E \in \mathcal{M} \text{ and } F \subset N \text{ for some } N \in \mathcal{N}\}$. Then $\overline{\mathcal{M}}$ is a σ-algebra, and there is a unique extension $\overline{\mu}$ of μ to a complete measure on $\overline{\mathcal{M}}$.*

Proof. Since $\mathcal{M}$ and $\mathcal{N}$ are closed under countable unions, so is $\overline{\mathcal{M}}$. If $E \cup F \in \overline{\mathcal{M}}$ where $E \in \mathcal{M}$ and $F \subset N \in \mathcal{N}$, we can assume that $E \cap N = \varnothing$ (otherwise, replace F and N by $F \setminus E$ and $N \setminus E$). Then $E \cup F = (E \cup N) \cap (N^c \cup F)$, so $(E \cup F)^c = (E \cup N)^c \cup (N \setminus F)$. But $(E \cup N)^c \in \mathcal{M}$ and $N \setminus F \subset N$, so that $(E \cup F)^c \in \overline{\mathcal{M}}$. Thus $\overline{\mathcal{M}}$ is a σ-algebra.

If $E \cup F \in \overline{\mathcal{M}}$ as above, we set $\overline{\mu}(E \cup F) = \mu(E)$. This is well defined, since if $E_1 \cup F_1 = E_2 \cup F_2$ where $F_j \subset N_j \in \mathcal{N}$, then $E_1 \subset E_2 \cup N_2$ and so $\mu(E_1) \leq \mu(E_2) + \mu(N_2) = \mu(E_2)$, and likewise $\mu(E_2) \leq \mu(E_1)$. It is easily verified that $\overline{\mu}$ is a complete measure on $\overline{\mathcal{M}}$, and that $\overline{\mu}$ is the only measure on $\overline{\mathcal{M}}$ that extends μ; details are left to the reader (Exercise 6). ∎

6. Complete the proof of Theorem 1.9.

7. If $\mu_1, \ldots, \mu_n$ are measures on $(X, \mathcal{M})$ and $a_1, \ldots, a_n \in [0, \infty)$, then $\sum_1^n a_j \mu_j$ is a measure on $(X, \mathcal{M})$.

8. If $(X, \mathcal{M}, \mu)$ is a measure space and $\{E_j\}_1^\infty \subset \mathcal{M}$, then $\mu(\liminf E_j) \leq \liminf \mu(E_j)$. Also, $\mu(\limsup E_j) \geq \limsup \mu(E_j)$ provided that $\mu(\bigcup_1^\infty E_j) < \infty$.

9. If $(X, \mathcal{M}, \mu)$ is a measure space and $E, F \in \mathcal{M}$, then $\mu(E) + \mu(F) = \mu(E \cup F) + \mu(E \cap F)$.

10. Given a measure space $(X, \mathcal{M}, \mu)$ and $E \in \mathcal{M}$, define $\mu_E(A) = \mu(A \cap E)$ for $A \in \mathcal{M}$. Then μ_E is a measure.

11. A finitely additive measure μ is a measure iff it is continuous from below as in Theorem 1.8c. If $\mu(X) < \infty$, μ is a measure iff it is continuous from above as in Theorem 1.8d.

12. Let $(X, \mathcal{M}, \mu)$ be a finite measure space.

- **a.** If $E, F \in \mathcal{M}$ and $\mu(E \Delta F) = 0$, then $\mu(E) = \mu(F)$.
- **b.** Say that $E \sim F$ if $\mu(E \Delta F) = 0$; then $\sim$ is an equivalence relation on $\mathcal{M}$.
- **c.** For $E, F \in \mathcal{M}$, define $\rho(E, F) = \mu(E \Delta F)$. Then $\rho(E, G) \leq \rho(E, F) + \rho(F, G)$, and hence ρ defines a metric on the space $\mathcal{M}/\sim$ of equivalence classes.

13. Every σ-finite measure is semifinite.

14. If μ is a semifinite measure and $\mu(E) = \infty$, for any $C > 0$ there exists $F \subset E$ with $C < \mu(F) < \infty$.

15. Given a measure μ on $(X, \mathcal{M})$, define μ_0 on $\mathcal{M}$ by $\mu_0(E) = \sup\{\mu(F) : F \subset E \text{ and } \mu(F) < \infty\}$.

- **a.** μ_0 is a semifinite measure. It is called the **semifinite part** of μ.
- **b.** If μ is semifinite, then $\mu = \mu_0$. (Use Exercise 14.)

c. There is a measure ν on $\mathcal{M}$ (in general, not unique) which assumes only the values 0 and ∞ such that $\mu = \mu_0 + \nu$.

16. Let $(X, \mathcal{M}, \mu)$ be a measure space. A set $E \subset X$ is called **locally measurable** if $E \cap A \in \mathcal{M}$ for all $A \in \mathcal{M}$ such that $\mu(A) < \infty$. Let $\widetilde{\mathcal{M}}$ be the collection of all locally measurable sets. Clearly $\mathcal{M} \subset \widetilde{\mathcal{M}}$; if $\mathcal{M} = \widetilde{\mathcal{M}}$, then μ is called **saturated**.

a. If μ is σ-finite, then μ is saturated.
b. $\widetilde{\mathcal{M}}$ is a σ-algebra.
c. Define $\widetilde{\mu}$ on $\widetilde{\mathcal{M}}$ by $\widetilde{\mu}(E) = \mu(E)$ if $E \in \mathcal{M}$ and $\widetilde{\mu}(E) = \infty$ otherwise. Then $\widetilde{\mu}$ is a saturated measure on $\widetilde{\mathcal{M}}$, called the **saturation** of μ.
d. If μ is complete, so is $\widetilde{\mu}$.
e. Suppose that μ is semifinite. For $E \in \widetilde{\mathcal{M}}$, define $\underline{\mu}(E) = \sup\{\mu(A) : A \in \mathcal{M} \text{ and } A \subset E\}$. Then $\underline{\mu}$ is a saturated measure on $\widetilde{\mathcal{M}}$ that extends μ.
f. Let X_1, X_2 be disjoint uncountable sets, $X = X_1 \cup X_2$, and $\mathcal{M}$ the σ-algebra of countable or co-countable sets in X. Let μ_0 be counting measure on $\mathcal{P}(X_1)$, and define μ on $\mathcal{M}$ by $\mu(E) = \mu_0(E \cap X_1)$. Then μ is a measure on $\mathcal{M}$, $\widetilde{\mathcal{M}} = \mathcal{P}(X)$, and in the notation of parts (c) and (e), $\widetilde{\mu} \neq \underline{\mu}$.

Exercises

17. If μ^* is an outer measure on X and $\{A_j\}_1^\infty$ is a sequence of disjoint μ^*-measurable sets, then $\mu^*(E \cap (\bigcup_1^\infty A_j)) = \sum_1^\infty \mu^*(E \cap A_j)$ for any $E \subset X$.

18. Let $\mathcal{A} \subset \mathcal{P}(X)$ be an algebra, $\mathcal{A}_\sigma$ the collection of countable unions of sets in $\mathcal{A}$, and $\mathcal{A}_{\sigma\delta}$ the collection of countable intersections of sets in $\mathcal{A}_\sigma$. Let μ_0 be a premeasure on $\mathcal{A}$ and μ^* the induced outer measure.

a. For any $E \subset X$ and $\epsilon > 0$ there exists $A \in \mathcal{A}_\sigma$ with $E \subset A$ and $\mu^*(A) \leq \mu^*(E) + \epsilon$.
b. If $\mu^*(E) < \infty$, then E is μ^*-measurable iff there exists $B \in \mathcal{A}_{\sigma\delta}$ with $E \subset B$ and $\mu^*(B \setminus E) = 0$.
c. If μ_0 is σ-finite, the restriction $\mu^*(E) < \infty$ in (b) is superfluous.

19. Let μ^* be an outer measure on X induced from a finite premeasure μ_0. If $E \subset X$, define the **inner measure** of E to be $\mu_*(E) = \mu_0(X) - \mu^*(E^c)$. Then E is μ^*-measurable iff $\mu^*(E) = \mu_*(E)$. (Use Exercise 18.)

20. Let μ^* be an outer measure on X, $\mathcal{M}^*$ the σ-algebra of μ^*-measurable sets, $\overline{\mu} = \mu^*|\mathcal{M}^*$, and μ^+ the outer measure induced by $\overline{\mu}$ as in (1.12) (with $\overline{\mu}$ and $\mathcal{M}^*$ replacing μ_0 and $\mathcal{A}$).

a. If $E \subset X$, we have $\mu^*(E) \leq \mu^+(E)$, with equality iff there exists $A \in \mathcal{M}^*$ with $A \supset E$ and $\mu^*(A) = \mu^*(E)$.
b. If μ^* is induced from a premeasure, then $\mu^* = \mu^+$. (Use Exercise 18a.)
c. If $X = \{0, 1\}$, there exists an outer measure μ^* on X such that $\mu^* \neq \mu^+$.

21. Let μ^* be an outer measure induced from a premeasure and $\overline{\mu}$ the restriction of μ^* to the μ^*-measurable sets. Then $\overline{\mu}$ is saturated. (Use Exercise 18.)

22. Let $(X, \mathcal{M}, \mu)$ be a measure space, μ^* the outer measure induced by μ according to (1.12), $\mathcal{M}^*$ the σ-algebra of μ^*-measurable sets, and $\overline{\mu} = \mu^*|\mathcal{M}^*$.

a. If μ is σ-finite, then $\overline{\mu}$ is the completion of μ. (Use Exercise 18.)
b. In general, $\overline{\mu}$ is the saturation of the completion of μ. (See Exercises 16 and 21.)

23. Let $\mathcal{A}$ be the collection of finite unions of sets of the form $(a, b] \cap \mathbb{Q}$ where $-\infty \leq a < b \leq \infty$.

a. $\mathcal{A}$ is an algebra on $\mathbb{Q}$. (Use Proposition 1.7.)
b. The σ-algebra generated by $\mathcal{A}$ is $\mathcal{P}(\mathbb{Q})$.
c. Define μ_0 on $\mathcal{A}$ by $\mu_0(\varnothing) = 0$ and $\mu_0(A) = \infty$ for $A \neq \varnothing$. Then μ_0 is a premeasure on $\mathcal{A}$, and there is more than one measure on $\mathcal{P}(\mathbb{Q})$ whose restriction to $\mathcal{A}$ is μ_0.

24. Let μ be a finite measure on $(X, \mathcal{M})$, and let μ^* be the outer measure induced by μ. Suppose that $E \subset X$ satisfies $\mu^*(E) = \mu^*(X)$ (but not that $E \in \mathcal{M}$).

a. If $A, B \in \mathcal{M}$ and $A \cap E = B \cap E$, then $\mu(A) = \mu(B)$.
b. Let $\mathcal{M}_E = \{A \cap E : A \in \mathcal{M}\}$, and define the function ν on $\mathcal{M}_E$ defined by $\nu(A \cap E) = \mu(A)$ (which makes sense by (a)). Then $\mathcal{M}_E$ is a σ-algebra on E and ν is a measure on $\mathcal{M}_E$.

Exercises

25. Complete the proof of Theorem 1.19.

26. Prove Proposition 1.20. (Use Theorem 1.18.)

27. Prove Proposition 1.22a. (Show that if $x, y \in C$ and $x < y$, there exists $z \notin C$ such that $x < z < y$.)

28. Let F be increasing and right continuous, and let μ_F be the associated measure. Then $\mu_F(\{a\}) = F(a) - F(a-)$, $\mu_F([a, b)) = F(b-) - F(a-)$, $\mu_F([a, b]) = F(b) - F(a-)$, and $\mu_F((a, b)) = F(b-) - F(a)$.

29. Let E be a Lebesgue measurable set.

a. If $E \subset N$ where N is the nonmeasurable set described in §1.1, then $m(E) = 0$.
b. If $m(E) > 0$, then E contains a nonmeasurable set. (It suffices to assume $E \subset [0, 1]$. In the notation of §1.1, $E = \bigcup_{r \in R} E \cap N_r$.)

1.19 Theorem. *If $E \subset \mathbb{R}$, the following are equivalent.*
a. $E \in \mathcal{M}_\mu$.
b. $E = V \setminus N_1$ *where V is a G_δ set and* $\mu(N_1) = 0$.
c. $E = H \cup N_2$ *where H is an F_σ set and* $\mu(N_2) = 0$.

1.20 Proposition. *If $E \in \mathcal{M}_\mu$ and $\mu(E) < \infty$, then for every $\epsilon > 0$ there is a set A that is a finite union of open intervals such that $\mu(E \triangle A) < \epsilon$.*

1.22 Proposition. *Let C be the Cantor set.*
a. C is compact, nowhere dense, and totally disconnected (i.e., the only connected subsets of C are single points). Moreover, C has no isolated points.
b. $m(C) = 0$.
c. $\text{card}(C) = \mathfrak{c}$.

30. If $E \in \mathcal{L}$ and $m(E) > 0$, for any $\alpha < 1$ there is an open interval I such that $m(E \cap I) > \alpha m(I)$.

31. If $E \in \mathcal{L}$ and $m(E) > 0$, the set $E - E = \{x - y : x, y \in E\}$ contains an interval centered at 0. (If I is as in Exercise 30 with $\alpha > \frac{3}{4}$, then $E - E$ contains $(-\frac{1}{2}m(I), \frac{1}{2}m(I))$.)

32. Suppose $\{\alpha_j\}_1^\infty \subset (0, 1)$.

a. $\prod_1^\infty (1 - \alpha_j) > 0$ iff $\sum_1^\infty \alpha_j < \infty$. (Compare $\sum_1^\infty \log(1 - \alpha_j)$ to $\sum \alpha_j$.)
b. Given $\beta \in (0, 1)$, exhibit a sequence $\{\alpha_j\}$ such that $\prod_1^\infty (1 - \alpha_j) = \beta$.

33. There exists a Borel set $A \subset [0, 1]$ such that $0 < m(A \cap I) < m(I)$ for every subinterval I of $[0, 1]$. (Hint: Every subinterval of $[0, 1]$ contains Cantor-type sets of positive measure.)

Ch. 2

In Exercises 1–7, $(X, \mathcal{M})$ is a measurable space.

1. Let $f : X \to \overline{\mathbb{R}}$ and $Y = f^{-1}(\mathbb{R})$. Then f is measurable iff $f^{-1}(\{-\infty\}) \in \mathcal{M}$, $f^{-1}(\{\infty\}) \in \mathcal{M}$, and f is measurable on Y.

2. Suppose $f, g : X \to \overline{\mathbb{R}}$ are measurable.
a. fg is measurable (where $0 \cdot (\pm\infty) = 0$).
b. Fix $a \in \overline{\mathbb{R}}$ and define $h(x) = a$ if $f(x) = -g(x) = \pm\infty$ and $h(x) = f(x) + g(x)$ otherwise. Then h is measurable.

3. If $\{f_n\}$ is a sequence of measurable functions on X, then $\{x : \lim f_n(x) \text{ exists}\}$ is a measurable set.

4. If $f : X \to \overline{\mathbb{R}}$ and $f^{-1}((r, \infty]) \in \mathcal{M}$ for each $r \in \mathbb{Q}$, then f is measurable.

5. If $X = A \cup B$ where $A, B \in \mathcal{M}$, a function f on X is measurable iff f is measurable on A and on B.

6. The supremum of an uncountable family of measurable $\overline{\mathbb{R}}$-valued functions on X can fail to be measurable (unless the σ-algebra $\mathcal{M}$ is very special).

7. Suppose that for each $\alpha \in \mathbb{R}$ we are given a set $E_\alpha \in \mathcal{M}$ such that $E_\alpha \subset E_\beta$ whenever $\alpha < \beta$, $\bigcup_{\alpha \in \mathbb{R}} E_\alpha = X$, and $\bigcap_{\alpha \in \mathbb{R}} E_\alpha = \varnothing$. Then there is a measurable function $f : X \to \mathbb{R}$ such that $f(x) \le \alpha$ on E_α and $f(x) \ge \alpha$ on E_α^c for every α. (Use Exercise 4.)

8. If $f : \mathbb{R} \to \mathbb{R}$ is monotone, then f is Borel measurable.

9. Let $f : [0, 1] \to [0, 1]$ be the Cantor function (§1.5), and let $g(x) = f(x) + x$.
a. g is a bijection from $[0, 1]$ to $[0, 2]$, and $h = g^{-1}$ is continuous from [0,2] to [0,1].
b. If C is the Cantor set, $m(g(C)) = 1$.
c. By Exercise 29 of Chapter 1, $g(C)$ contains a Lebesgue nonmeasurable set A. Let $B = g^{-1}(A)$. Then B is Lebesgue measurable but not Borel.

8. If $f : \mathbb{R} \to \mathbb{R}$ is monotone, then f is Borel measurable.

9. Let $f : [0, 1] \to [0, 1]$ be the Cantor function (§1.5), and let $g(x) = f(x) + x$.
a. g is a bijection from $[0, 1]$ to $[0, 2]$, and $h = g^{-1}$ is continuous from [0,2] to [0,1].
b. If C is the Cantor set, $m(g(C)) = 1$.
c. By Exercise 29 of Chapter 1, $g(C)$ contains a Lebesgue nonmeasurable set A. Let $B = g^{-1}(A)$. Then B is Lebesgue measurable but not Borel.

Exercises

12. Prove Proposition 2.20. (See Proposition 0.20, where a special case is proved.)

13. Suppose $\{f_n\} \subset L^+$, $f_n \to f$ pointwise, and $\int f = \lim \int f_n < \infty$. Then $\int_E f = \lim \int_E f_n$ for all $E \in \mathcal{M}$. However, this need not be true if $\int f = \lim \int f_n = \infty$.

14. If $f \in L^+$, let $\lambda(E) = \int_E f\,d\mu$ for $E \in \mathcal{M}$. Then λ is a measure on $\mathcal{M}$, and for any $g \in L^+$, $\int g\,d\lambda = \int fg\,d\mu$. (First suppose that g is simple.)

15. If $\{f_n\} \subset L^+$, f_n decreases pointwise to f, and $\int f_1 < \infty$, then $\int f = \lim \int f_n$.

16. If $f \in L^+$ and $\int f < \infty$, for every $\epsilon > 0$ there exists $E \in \mathcal{M}$ such that $\mu(E) < \infty$ and $\int_E f > (\int f) - \epsilon$.

17. Assume Fatou's lemma and deduce the monotone convergence theorem from it.

Exercises

18. Fatou's lemma remains valid if the hypothesis that $f_n \in L^+$ is replaced by the hypothesis that f_n is measurable and $f_n \ge -g$ where $g \in L^+ \cap L^1$. What is the analogue of Fatou's lemma for nonpositive functions?

19. Suppose $\{f_n\} \subset L^1(\mu)$ and $f_n \to f$ uniformly.
a. If $\mu(X) < \infty$, then $f \in L^1(\mu)$ and $\int f_n \to \int f$.
b. If $\mu(X) = \infty$, the conclusions of (a) can fail. (Find examples on $\mathbb{R}$ with Lebesgue measure.)

20. (A generalized Dominated Convergence Theorem) If $f_n, g_n, f, g \in L^1$, $f_n \to f$ and $g_n \to g$ a.e., $|f_n| \le g_n$, and $\int g_n \to \int g$, then $\int f_n \to \int f$. (Rework the proof of the dominated convergence theorem.)

21. Suppose $f_n, f \in L^1$ and $f_n \to f$ a.e. Then $\int |f_n - f| \to 0$ iff $\int |f_n| \to \int |f|$. (Use Exercise 20.)

22. Let μ be counting measure on $\mathbb{N}$. Interpret Fatou's lemma and the monotone and dominated convergence theorems as statements about infinite series.

23. Given a bounded function $f : [a, b] \to \mathbb{R}$, let

$$H(x) = \lim_{\delta \to 0} \sup_{|y-x| \le \delta} f(y), \qquad h(x) = \lim_{\delta \to 0} \inf_{|y-x| \le \delta} f(y).$$

2.20 Proposition. *If $f \in L^+$ and $\int f < \infty$, then $\{x : f(x) = \infty\}$ is a null set and $\{x : f(x) > 0\}$ is σ-finite.*

The proof is left to the reader (Exercise 12).

23. Given a bounded function $f : [a,b] \to \mathbb{R}$, let

$$H(x) = \lim_{\delta \to 0} \sup_{|y-x| \le \delta} f(y), \qquad h(x) = \lim_{\delta \to 0} \inf_{|y-x| \le \delta} f(y).$$

Prove Theorem 2.28b by establishing the following lemmas:

a. $H(x) = h(x)$ iff f is continuous at x.

b. In the notation of the proof of Theorem 2.28a, $H = G$ a.e. and $h = g$ a.e. Hence H and h are Lebesgue measurable, and $\int_{[a,b]} H\, dm = \overline{I}_a^b(f)$ and $\int_{[a,b]} h\, dm = \underline{I}_a^b(f)$.

24. Let $(X, \mathcal{M}, \mu)$ be a measure space with $\mu(X) < \infty$, and let $(X, \overline{\mathcal{M}}, \overline{\mu})$ be its completion. Suppose $f : X \to \mathbb{R}$ is bounded. Then f is $\overline{\mathcal{M}}$-measurable (and hence in $L^1(\overline{\mu})$) iff there exist sequences $\{\phi_n\}$ and $\{\psi_n\}$ of $\mathcal{M}$-measurable simple functions such that $\phi_n \le f \le \psi_n$ and $\int (\psi_n - \phi_n)\, d\mu < n^{-1}$. In this case, $\lim \int \phi_n\, d\mu = \lim \int \psi_n\, d\mu = \int f\, d\overline{\mu}$.

25. Let $f(x) = x^{-1/2}$ if $0 < x < 1$, $f(x) = 0$ otherwise. Let $\{r_n\}_1^\infty$ be an enumeration of the rationals, and set $g(x) = \sum_1^\infty 2^{-n} f(x - r_n)$.

a. $g \in L^1(m)$, and in particular $g < \infty$ a.e.

b. g is discontinuous at every point and unbounded on every interval, and it remains so after any modification on a Lebesgue null set.

c. $g^2 < \infty$ a.e., but g^2 is not integrable on any interval.

26. If $f \in L^1(m)$ and $F(x) = \int_{-\infty}^x f(t)\, dt$, then F is continuous on $\mathbb{R}$.

27. Let $f_n(x) = ae^{-nax} - be^{-nbx}$ where $0 < a < b$.

a. $\sum_1^\infty \int_0^\infty |f_n(x)|\, dx = \infty$.

b. $\sum_1^\infty \int_0^\infty f_n(x)\, dx = 0$.

c. $\sum_1^\infty f_n \in L^1([0,\infty), m)$, and $\int_0^\infty \sum_1^\infty f_n(x)\, dx = \log(b/a)$.

28. Compute the following limits and justify the calculations:

a. $\lim_{n\to\infty} \int_0^\infty (1 + (x/n))^{-n} \sin(x/n)\, dx$.

b. $\lim_{n\to\infty} \int_0^1 (1 + nx^2)(1 + x^2)^{-n}\, dx$.

c. $\lim_{n\to\infty} \int_0^\infty n \sin(x/n)[x(1 + x^2)]^{-1}\, dx$.

d. $\lim_{n\to\infty} \int_a^\infty n(1 + n^2x^2)^{-1}\, dx$. (The answer depends on whether $a > 0$, $a = 0$, or $a < 0$. How does this accord with the various convergence theorems?)

29. Show that $\int_0^\infty x^n e^{-x}\, dx = n!$ by differentiating the equation $\int_0^\infty e^{-tx}\, dx = 1/t$. Similarly, show that $\int_{-\infty}^\infty x^{2n} e^{-x^2}\, dx = (2n)!\sqrt{\pi}/4^n n!$ by differentiating the equation $\int_{-\infty}^\infty e^{-tx^2}\, dx = \sqrt{\pi/t}$ (see Proposition 2.53).

30. Show that $\lim_{k\to\infty} \int_0^k x^n (1 - k^{-1}x)^k\, dx = n!$.

31. Derive the following formulas by expanding part of the integrand into an infinite series and justifying the term-by-term integration. Exercise 29 may be useful. (*Note:* In (d) and (e), term-by-term integration works, and the resulting series converges, only for $a > 1$, but the formulas as stated are actually valid for all $a > 0$.)

a. For $a > 0$, $\int_{-\infty}^\infty e^{-x^2} \cos ax\, dx = \sqrt{\pi} e^{-a^2/4}$.

b. For $a > -1$, $\int_0^1 x^a (1 - x)^{-1} \log x\, dx = \sum_1^\infty (a + k)^{-2}$.

c. For $a > 1$, $\int_0^\infty x^{a-1}(e^x - 1)^{-1}\, dx = \Gamma(a)\zeta(a)$, where $\zeta(a) = \sum_1^\infty n^{-a}$.

d. For $a > 1$, $\int_0^\infty e^{-ax} x^{-1} \sin x\, dx = \arctan(a^{-1})$.

e. For $a > 1$, $\int_0^\infty e^{-ax} J_0(x)\, dx = (s^2 + 1)^{-1/2}$, where $J_0(x) = \sum_0^\infty (-1)^n x^{2n}/4^n (n!)^2$ is the Bessel function of order zero.

32. Suppose $\mu(X) < \infty$. If f and g are complex-valued measurable functions on X, define

$$\rho(f, g) = \int \frac{|f - g|}{1 + |f - g|}\, d\mu.$$

Then ρ is a metric on the space of measurable functions if we identify functions that are equal a.e., and $f_n \to f$ with respect to this metric iff $f_n \to f$ in measure.

33. If $f_n \ge 0$ and $f_n \to f$ in measure, then $\int f \le \liminf \int f_n$.

34. Suppose $|f_n| \le g \in L^1$ and $f_n \to f$ in measure.

a. $\int f = \lim \int f_n$.

b. $f_n \to f$ in L^1.

35. $f_n \to f$ in measure iff for every $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that $\mu(\{x : |f_n(x) - f(x)| \ge \epsilon\}) < \epsilon$ for $n \ge N$.

36. If $\mu(E_n) < \infty$ for $n \in \mathbb{N}$ and $\chi_{E_n} \to f$ in L^1, then f is (a.e. equal to) the characteristic function of a measurable set.

37. Suppose that f_n and f are measurable complex-valued functions and $\phi : \mathbb{C} \to \mathbb{C}$.

a. If ϕ is continuous and $f_n \to f$ a.e., then $\phi \circ f_n \to \phi \circ f$ a.e.

b. If ϕ is uniformly continuous and $f_n \to f$ uniformly, almost uniformly, or in measure, then $\phi \circ f_n \to \phi \circ f$ uniformly, almost uniformly, or in measure, respectively.

c. There are counterexamples when the continuity assumptions on ϕ are not satisfied.

38. Suppose $f_n \to f$ in measure and $g_n \to g$ in measure.

a. $f_n + g_n \to f + g$ in measure.

b. $f_n g_n \to fg$ in measure if $\mu(X) < \infty$, but not necessarily if $\mu(X) = \infty$.

39. If $f_n \to f$ almost uniformly, then $f_n \to f$ a.e. and in measure.

40. In Egoroff's theorem, the hypothesis "$\mu(X) < \infty$" can be replaced by "$|f_n| \leq g$ for all n, where $g \in L^1(\mu)$."

41. If μ is σ-finite and $f_n \to f$ a.e., there exist measurable $E_1, E_2, \ldots \subset X$ such that $\mu((\bigcup_1^\infty E_j)^c) = 0$ and $f_n \to f$ uniformly on each E_j.

42. Let μ be counting measure on $\mathbb{N}$. Then $f_n \to f$ in measure iff $f_n \to f$ uniformly.

43. Suppose that $\mu(X) < \infty$ and $f : X \times [0, 1] \to \mathbb{C}$ is a function such that $f(\cdot, y)$ is measurable for each $y \in [0, 1]$ and $f(x, \cdot)$ is continuous for each $x \in X$.

a. If $0 < \epsilon, \delta < 1$ then $E_{\epsilon,\delta} = \{x : |f(x, y) - f(x, 0)| \leq \epsilon \text{ for all } y < \delta\}$ is measurable.

b. For any $\epsilon > 0$ there is a set $E \subset X$ such that $\mu(E) < \epsilon$ and $f(\cdot, y) \to f(\cdot, 0)$ uniformly on E^c as $y \to 0$.

44. **(Lusin's Theorem)** If $f : [a, b] \to \mathbb{C}$ is Lebesgue measurable and $\epsilon > 0$, there is a compact set $E \subset [a, b]$ such that $\mu(E^c) < \epsilon$ and $f|E$ is continuous. (Use Egoroff's theorem and Theorem 2.26.)

Exercises

45. If $(X_j, \mathcal{M}_j)$ is a measurable space for $j = 1, 2, 3$, then $\bigotimes_1^3 \mathcal{M}_j = (\mathcal{M}_1 \otimes \mathcal{M}_2) \otimes \mathcal{M}_3$. Moreover, if μ_j is a σ-finite measure on $(X_j, \mathcal{M}_j)$, then $\mu_1 \times \mu_2 \times \mu_3 = (\mu_1 \times \mu_2) \times \mu_3$.

46. Let $X = Y = [0, 1]$, $\mathcal{M} = \mathcal{N} = \mathcal{B}_{[0,1]}$, $\mu =$ Lebesgue measure, and $\nu =$ counting measure. If $D = \{(x, x) : x \in [0, 1]\}$ is the diagonal in $X \times Y$, then $\iint \chi_D \, d\mu \, d\nu$, $\iint \chi_D \, d\nu \, d\mu$, and $\int \chi_D \, d(\mu \times \nu)$ are all unequal. (To compute $\int \chi_D \, d(\mu \times \nu) = \mu \times \nu(D)$, go back to the definition of $\mu \times \nu$.)

47. Let $X = Y$ be an uncountable linearly ordered set such that for each $x \in X$, $\{y \in X : y < x\}$ is countable. (Example: the set of countable ordinals.) Let $\mathcal{M} = \mathcal{N}$ be the σ-algebra of countable or co-countable sets, and let $\mu = \nu$ be defined on $\mathcal{M}$ by $\mu(A) = 0$ if A is countable and $\mu(A) = 1$ if A is co-countable. Let $E = \{(x, y) \in X \times X : y < x\}$. Then E_x and E^y are measurable for all x, y, and $\iint \chi_E \, d\mu \, d\nu$ and $\iint \chi_E \, d\nu \, d\mu$ exist but are not equal. (If one believes in the continuum hypothesis, one can take $X = [0, 1]$ [with a nonstandard ordering] and thus obtain a set $E \subset [0, 1]^2$ such that E_x is countable and E^y is co-countable [in particular, Borel] for all x, y, but E is not Lebesgue measurable.)

48. Let $X = Y = \mathbb{N}$, $\mathcal{M} = \mathcal{N} = \mathcal{P}(\mathbb{N})$, $\mu = \nu =$ counting measure. Define $f(m, n) = 1$ if $m = n$, $f(m, n) = -1$ if $m = n + 1$, and $f(m, n) = 0$ otherwise. Then $\int |f| \, d(\mu \times \nu) = \infty$, and $\iint f \, d\mu \, d\nu$ and $\iint f \, d\nu \, d\mu$ exist and are unequal.

49. Prove Theorem 2.39 by using Theorem 2.37 and Proposition 2.12 together with the following lemmas.

a. If $E \in \mathcal{M} \times \mathcal{N}$ and $\mu \times \nu(E) = 0$, then $\nu(E_x) = \mu(E^y) = 0$ for a.e. x and y.

b. If f is $\mathcal{L}$-measurable and $f = 0$ λ-a.e., then f_x and f^y are integrable for a.e. x and y, and $\int f_x \, d\nu = \int f^y \, d\mu = 0$ for a.e. x and y. (Here the completeness of μ and ν is needed.)

50. Suppose $(X, \mathcal{M}, \mu)$ is a σ-finite measure space and $f \in L^+(X)$. Let

$$G_f = \{(x,y) \in X \times [0,\infty] : y \le f(x)\}.$$

Then G_f is $\mathcal{M} \times \mathcal{B}_{\mathbb{R}}$-measurable and $\mu \times m(G_f) = \int f \, d\mu$; the same is also true if the inequality $y \le f(x)$ in the definition of G_f is replaced by $y < f(x)$. (To show measurability of G_f, note that the map $(x,y) \mapsto f(x) - y$ is the composition of $(x,y) \mapsto (f(x), y)$ and $(z,y) \mapsto z - y$.) This is the definitive statement of the familiar theorem from calculus, "the integral of a function is the area under its graph."

51. Let $(X, \mathcal{M}, \mu)$ and $(Y, \mathcal{N}, \nu)$ be arbitrary measure spaces (not necessarily σ-finite).

a. If $f : X \to \mathbb{C}$ is $\mathcal{M}$-measurable, $g : Y \to \mathbb{C}$ is $\mathcal{N}$-measurable, and $h(x,y) = f(x)g(y)$, then h is $\mathcal{M} \otimes \mathcal{N}$-measurable.

b. If $f \in L^1(\mu)$ and $g \in L^1(\nu)$, then $h \in L^1(\mu \times \nu)$ and $\int h \, d(\mu \times \nu) = [\int f \, d\mu][\int g \, d\nu]$.

52. The Fubini-Tonelli theorem is valid when $(X, \mathcal{M}, \mu)$ is an arbitrary measure space and Y is a countable set, $\mathcal{N} = \mathcal{P}(Y)$, and ν is counting measure on Y. (Cf. Theorems 2.15 and 2.25.)

Exercises

53. Fill in the details of the proof of Theorem 2.41.

54. How much of Theorem 2.44 remains valid if T is not invertible?

2.44 Theorem. *Suppose $T \in GL(n, \mathbb{R})$.*

a. If f is a Lebesgue measurable function on $\mathbb{R}^n$, so is $f \circ T$. If $f \ge 0$ or $f \in L^1(m)$, then

$$\int f(x)\, dx = |\det T| \int f \circ T(x)\, dx. \tag{2.45}$$

b. If $E \in \mathcal{L}^n$, then $T(E) \in \mathcal{L}^n$ and $m(T(E)) = |\det T| m(E)$.

2.41 Theorem. *If $f \in L^1(m)$ and $\epsilon > 0$, there is a simple function $\phi = \sum_1^N a_j \chi_{R_j}$, where each R_j is a product of intervals, such that $\int |f - \phi| < \epsilon$, and there is a continuous function g that vanishes outside a bounded set such that $\int |f - g| < \epsilon$.*

55. Let $E = [0,1] \times [0,1]$. Investigate the existence and equality of $\int_E f \, dm^2$, $\int_0^1 \int_0^1 f(x,y)\, dx\, dy$, and $\int_0^1 \int_0^1 f(x,y)\, dy\, dx$ for the following f.

a. $f(x,y) = (x^2 - y^2)(x^2 + y^2)^{-2}$.

b. $f(x,y) = (1 - xy)^{-a}$ $(a > 0)$.

c. $f(x,y) = (x - \frac{1}{2})^{-3}$ if $0 < y < |x - \frac{1}{2}|$, $f(x,y) = 0$ otherwise.

56. If f is Lebesgue integrable on $(0,a)$ and $g(x) = \int_x^a t^{-1} f(t)\, dt$, then g is integrable on $(0,a)$ and $\int_0^a g(x)\, dx = \int_0^a f(x)\, dx$.

57. Show that $\int_0^\infty e^{-sx} x^{-1} \sin x \, dx = \arctan(s^{-1})$ for $s > 0$ by integrating $e^{-sxy} \sin x$ with respect to x and y. (It may be useful to recall that $\tan(\frac{\pi}{2} - \theta) = (\tan\theta)^{-1}$. Cf. Exercise 31d.)

58. Show that $\int e^{-sx} x^{-1} \sin^2 x \, dx = \frac{1}{4}\log(1 + 4s^{-2})$ for $s > 0$ by integrating $e^{-sx} \sin 2xy$ with respect to x and y.

59. Let $f(x) = x^{-1}\sin x$.

a. Show that $\int_0^\infty |f(x)|\, dx = \infty$.

b. Show that $\lim_{b \to \infty} \int_0^b f(x)\, dx = \frac{1}{2}\pi$ by integrating $e^{-xy} \sin x$ with respect to x and y. (In view of part (a), some care is needed in passing to the limit as $b \to \infty$.)

60. $\Gamma(x)\Gamma(y)/\Gamma(x+y) = \int_0^1 t^{x-1}(1-t)^{y-1}\, dt$ for $x, y > 0$. (Recall that Γ was defined in §2.3. Write $\Gamma(x)\Gamma(y)$ as a double integral and use the argument of the exponential as a new variable of integration.)

61. If f is continuous on $[0,\infty)$, for $\alpha > 0$ and $x \ge 0$ let

$$I_\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} f(t)\, dt.$$

$I_\alpha f$ is called the αth **fractional integral** of f.

a. $I_{\alpha+\beta} f = I_\alpha(I_\beta f)$ for all $\alpha, \beta > 0$. (Use Exercise 60.)

b. If $n \in \mathbb{N}$, $I_n f$ is an nth-order antiderivative of f.

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62. The measure σ on S^{n-1} is invariant under rotations.

63. The technique used to prove Proposition 2.54 can also be used to integrate any polynomial over S^{n-1}. In fact, suppose $f(x) = \prod_1^n x_j^{\alpha_j}$ ($\alpha_j \in \mathbb{N} \cup \{0\}$) is a monomial. Then $\int f\,d\sigma = 0$ if any α_j is odd, and if all α_j's are even,

$$\int f\,d\sigma = \frac{2\Gamma(\beta_1)\cdots\Gamma(\beta_n)}{\Gamma(\beta_1+\cdots+\beta_n)}, \quad \text{where } \beta_j = \frac{\alpha_j+1}{2}.$$

64. For which real values of a and b is $|x|^a \left|\log|x|\right|^b$ integrable over $\{x \in \mathbb{R}^n : |x| < \frac{1}{2}\}$? Over $\{x \in \mathbb{R}^n : |x| > 2\}$?

65. Define $G : \mathbb{R}^n \to \mathbb{R}^n$ by $G(r, \phi_1, \ldots, \phi_{n-2}, \theta) = (x_1, \ldots, x_n)$ where

$$x_1 = r\cos\phi_1, \quad x_2 = r\sin\phi_1\cos\phi_2, \quad x_3 = r\sin\phi_1\sin\phi_2\cos\phi_3, \ldots,$$
$$x_{n-1} = r\sin\phi_1\cdots\sin\phi_{n-2}\cos\theta, \quad x_n = r\sin\phi_1\cdots\sin\phi_{n-2}\sin\theta.$$

a. G maps $\mathbb{R}^n$ onto $\mathbb{R}^n$, and $|G(r, \phi_1, \ldots, \phi_{n-2}, \theta)| = |r|$.
b. $\det D_{(r,\phi_1,\ldots,\phi_{n-2},\theta)}G = r^{n-1}\sin^{n-2}\phi_1\sin^{n-3}\phi_2\cdots\sin\phi_{n-2}$.
c. Let $\Omega = (0,\infty)\times(0,\pi)^{n-2}\times(0,2\pi)$. Then $G|\Omega$ is a diffeomorphism and $m(\mathbb{R}^n \setminus G(\Omega)) = 0$.
d. Let $F(\phi_1, \ldots, \phi_{n-2}, \theta) = G(1, \phi_1, \ldots, \phi_{n-2}, \theta)$ and $\Omega' = (0,\pi)^{n-2}\times(0,2\pi)$. Then $(F|\Omega')^{-1}$ defines a coordinate system on S^{n-1} except on a σ-null set, and the measure σ is given in these coordinates by

$$d\sigma(\phi_1, \ldots \phi_{n-2}, \theta) = \sin^{n-2}\phi_1\sin^{n-3}\phi_2\cdots\sin\phi_{n-2}\,d\phi_1\cdots d\phi_{n-2}\,d\theta.$$

Exercises

1. Prove Proposition 3.1.

2. If ν is a signed measure, E is ν-null iff $|\nu|(E) = 0$. Also, if ν and μ are signed measures, $\nu \perp \mu$ iff $|\nu| \perp \mu$ iff $\nu^+ \perp \mu$ and $\nu^- \perp \mu$.

3. Let ν be a signed measure on $(X, \mathcal{M})$.
a. $L^1(\nu) = L^1(|\nu|)$.
b. If $f \in L^1(\nu)$, $|\int f\,d\nu| \le \int |f|\,d|\nu|$.
c. If $E \in \mathcal{M}$, $|\nu|(E) = \sup\{|\int_E f\,d\nu| : |f| \le 1\}$.

4. If ν is a signed measure and λ, μ are positive measures such that $\nu = \lambda - \mu$, then $\lambda \ge \nu^+$ and $\mu \ge \nu^-$.

5. If ν_1, ν_2 are signed measures that both omit the value $+\infty$ or $-\infty$, then $|\nu_1+\nu_2| \le |\nu_1| + |\nu_2|$. (Use Exercise 4.)

6. Suppose $\nu(E) = \int_E f\,d\mu$ where μ is a positive measure and f is an extended μ-integrable function. Describe the Hahn decompositions of ν and the positive, negative, and total variations of ν in terms of f and μ.

7. Suppose that ν is a signed measure on $(X, \mathcal{M})$ and $E \in \mathcal{M}$.
a. $\nu^+(E) = \sup\{\nu(F) : F \in \mathcal{M},\ F \subset E\}$ and $\nu^-(E) = -\inf\{\nu(F) : F \in \mathcal{M},\ F \subset E\}$.
b. $|\nu|(E) = \sup\{\sum_1^n |\nu(E_j)| : n \in \mathbb{N},\ E_1, \ldots, E_n$ are disjoint, and $\bigcup_1^n E_j = E\}$.

Exercises

8. $\nu \ll \mu$ iff $|\nu| \ll \mu$ iff $\nu^+ \ll \mu$ and $\nu^- \ll \mu$.

9. Suppose $\{\nu_j\}$ is a sequence of positive measures. If $\nu_j \perp \mu$ for all j, then $\sum_1^\infty \nu_j \perp \mu$; and if $\nu_j \ll \mu$ for all j, then $\sum_1^\infty \nu_j \ll \mu$.

10. Theorem 3.5 may fail when ν is not finite. (Consider $d\nu(x) = dx/x$ and $d\mu(x) = dx$ on $(0,1)$, or ν = counting measure and $\mu(E) = \sum_{n\in E} 2^{-n}$ on $\mathbb{N}$.)

11. Let μ be a positive measure. A collection of functions $\{f_\alpha\}_{\alpha\in A} \subset L^1(\mu)$ is called **uniformly integrable** if for every $\epsilon > 0$ there exists $\delta > 0$ such that $|\int_E f_\alpha\,d\mu| < \epsilon$ for all $\alpha \in A$ whenever $\mu(E) < \delta$.
a. Any finite subset of $L^1(\mu)$ is uniformly integrable.
b. If $\{f_n\}$ is a sequence in $L^1(\mu)$ that converges in the L^1 metric to $f \in L^1(\mu)$, then $\{f_n\}$ is uniformly integrable.

12. For $j = 1, 2$, let μ_j, ν_j be σ-finite measures on $(X_j, \mathcal{M}_j)$ such that $\nu_j \ll \mu_j$. Then $\nu_1 \times \nu_2 \ll \mu_1 \times \mu_2$ and

$$\frac{d(\nu_1\times\nu_2)}{d(\mu_1\times\mu_2)}(x_1, x_2) = \frac{d\nu_1}{d\mu_1}(x_1)\frac{d\nu_2}{d\mu_2}(x_2).$$

13. Let $X = [0,1]$, $\mathcal{M} = \mathcal{B}_{[0,1]}$, m = Lebesgue measure, and μ = counting measure on $\mathcal{M}$.
a. $m \ll \mu$ but $dm \neq f\,d\mu$ for any f.
b. μ has no Lebesgue decomposition with respect to m.

14. If ν is an arbitrary signed measure and μ is a σ-finite measure on $(X, \mathcal{M})$ such that $\nu \ll \mu$, there exists an extended μ-integrable function $f : X \to [-\infty, \infty]$ such that $d\nu = f\,d\mu$. Hints:
a. It suffices to assume that μ is finite and ν is positive.
b. With these assumptions, there exists $E \in \mathcal{M}$ that is σ-finite for ν such that $\mu(E) \ge \mu(F)$ for all sets F that are σ-finite for ν.
c. The Radon-Nikodym theorem applies on E. If $F \cap E = \varnothing$, then either $\nu(F) = \mu(F) = 0$ or $\mu(F) > 0$ and $|\nu(F)| = \infty$.

15. A measure μ on $(X, \mathcal{M})$ is called **decomposable** if there is a family $\mathcal{F} \subset \mathcal{M}$ with the following properties: (i) $\mu(F) < \infty$ for all $F \in \mathcal{F}$; (ii) the members of $\mathcal{F}$ are disjoint and their union is X; (iii) if $\mu(E) < \infty$ then $\mu(E) = \sum_{F\in\mathcal{F}} \mu(E \cap F)$; (iv) if $E \subset X$ and $E \cap F \in \mathcal{M}$ for all $F \in \mathcal{F}$ then $E \in \mathcal{M}$.
a. Every σ-finite measure is decomposable.
b. If μ is decomposable and ν is any signed measure on $(X, \mathcal{M})$ such that $\nu \ll \mu$, there exists a measurable $f : X \to [-\infty, \infty]$ such that $\nu(E) = \int_E f\,d\mu$ for any E that is σ-finite for μ, and $|f| < \infty$ on any $F \in \mathcal{F}$ that is σ-finite for ν. (Use Exercise 14 if ν is not σ-finite.)

16. Suppose that μ, ν are measures on $(X, \mathcal{M})$ with $\nu \ll \mu$, and let $\lambda = \mu + \nu$. If $f = d\nu/d\lambda$, then $0 \le f < 1$ μ-a.e. and $d\nu/d\mu = f/(1-f)$.

17. Let $(X, \mathcal{M}, \mu)$ be a σ-finite measure space, $\mathcal{N}$ a sub-σ-algebra of $\mathcal{M}$, and $\nu = \mu|\mathcal{N}$. If $f \in L^1(\mu)$, there exists $g \in L^1(\nu)$ (thus g is $\mathcal{N}$-measurable) such that $\int_E f\,d\mu = \int_E g\,d\nu$ for all $E \in \mathcal{N}$; if g' is another such function then $g = g'$ ν-a.e. (In probability theory, g is called the **conditional expectation** of f on $\mathcal{N}$.)

Exercises

18. Prove Proposition 3.13c.

19. If ν, μ are complex measures and λ is a positive measure, then $\nu \perp \mu$ iff $|\nu| \perp |\mu|$, and $\nu \ll \lambda$ iff $|\nu| \ll \lambda$.

20. If ν is a complex measure on $(X, \mathcal{M})$ and $\nu(X) = |\nu|(X)$, then $\nu = |\nu|$.

21. Let ν be a complex measure on $(X, \mathcal{M})$. If $E \in \mathcal{M}$, define

$$\mu_1(E) = \sup\Big\{\sum_1^n |\nu(E_j)| : n \in \mathbb{N},\ E_1, \ldots, E_n \text{ disjoint},\ E = \bigcup_1^n E_j\Big\},$$

$$\mu_2(E) = \sup\Big\{\sum_1^\infty |\nu(E_j)| : E_1, E_2, \ldots \text{ disjoint},\ E = \bigcup_1^\infty E_j\Big\},$$

$$\mu_3(E) = \sup\Big\{\Big|\int_E f\,d\mu\Big| : |f| \le 1\Big\}.$$

Then $\mu_1 = \mu_2 = \mu_3 = |\nu|$. (First show that $\mu_1 \le \mu_2 \le \mu_3$. To see that $\mu_3 = |\nu|$, let $f = d\nu/d|\nu|$ and apply Proposition 3.13. To see that $\mu_3 \le \mu_1$, approximate f by simple functions.)

3.13 Proposition. *Let ν be a complex measure on $(X, \mathcal{M})$.*
a. $|\nu(E)| \le |\nu|(E)$ *for all* $E \in \mathcal{M}$.
b. $\nu \ll |\nu|$, *and* $d\nu/d|\nu|$ *has absolute value* 1 $|\nu|$*-a.e.*
c. $L^1(\nu) = L^1(|\nu|)$, *and if* $f \in L^1(\nu)$, *then* $|\int f\,d\nu| \le \int |f|\,d|\nu|$.

Exercises

22. If $f \in L^1(\mathbb{R}^n)$, $f \neq 0$, there exist $C, R > 0$ such that $Hf(x) \ge C|x|^{-n}$ for $|x| > R$. Hence $m(\{x : Hf(x) > \alpha\}) \ge C'/\alpha$ when α is small, so the estimate in the maximal theorem is essentially sharp.

23. A useful variant of the Hardy-Littlewood maximal function is

$$H^*f(x) = \sup\left\{\frac{1}{m(B)}\int_B |f(y)|\,dy : B \text{ is a ball and } x \in B\right\}.$$

Show that $Hf \le H^*f \le 2^n Hf$.

24. If $f \in L^1_{\text{loc}}$ and f is continuous at x, then x is in the Lebesgue set of f.

25. If E is a Borel set in $\mathbb{R}^n$, the **density** $D_E(x)$ of E at x is defined as

$$D_E(x) = \lim_{r\to 0}\frac{m(E \cap B(r,x))}{m(B(r,x))},$$

whenever the limit exists.
a. Show that $D_E(x) = 1$ for a.e. $x \in E$ and $D_E(x) = 0$ for a.e. $x \in E^c$.
b. Find examples of E and x such that $D_E(x)$ is a given number $\alpha \in (0,1)$, or such that $D_E(x)$ does not exist.

26. If λ and μ are positive, mutually singular Borel measures on $\mathbb{R}^n$ and $\lambda + \mu$ is regular, then so are λ and μ.