Math 871 Fall 2013 Midterm 1 Review

Notes: You need to know complete definitions of the italicized words under Vocabulary below, along with the complete statements of the theorems paraphrased below which have a • next to their statements. You will need to be able to use all other theorems paraphrased below that are preceded by a ∘. If a theorem was stated in the exercises, its number is preceded by an E. Use of the text or your notes will not be allowed on this exam.

1. Set theory background (Sec. 2, 6, 7):
Vocabulary: *Finite, infinite, countable, image, preimage.*
Theorems:
- ∘ de Morgan's Laws: $A-(B\cup C)=(A-B)\cap(A-C)$ and $A-(B\cap C)=(A-B)\cup(A-C)$.
- ∘ E2.1, E2.2, E2.3: Preimages are nice! Images are sometimes.
- · 6.7: Tfae: B is finite, $\exists$ surjection $\{0,...,n\}\to B$, $\exists$ injection $B\to\{0,...,n\}$.
- ∘ 6.6, 6.8: Subsets, finite unions, and products of finite sets are finite.
- · Prop.: Tfae: B is infinite, $\exists$ surjection $B\to\mathbb{N}$, $\exists$ injection $\mathbb{N}\to B$.
- · 7.1: Tfae: B is countable, $\exists$ surjection $\mathbb{N}\to B$, $\exists$ injection $B\to\mathbb{N}$.
- ∘ 7.3, 7.5, 7.6: Subsets, countable unions, finite products of countable sets are countable.

2. Topologies, bases, and subbases (Sec. 12, 13):
Vocabulary: *Topology, topological space, open, finer, coarser, basis, topology generated by basis, subbasis, topology generated by subbasis.* Topologies: *discrete, indiscrete, finite complement, included point, excluded point, Euclidean, infinite ray,* line with two origins, inverse image, lower limit.
Theorems:
- ∘ 13.1: $\mathcal{T}(\mathcal{B})$ is the collection of unions of basis elements.
- • 13.3: $\mathcal{T}(\mathcal{B})\subseteq\mathcal{T}(\mathcal{B}')$ iff $\forall B\in\mathcal{B}$ and $x\in\mathcal{B}$, $\exists B'\in\mathcal{B}'$ with $x\in B'\subseteq B$.
- ∘ Lem.: f is continuous iff preimages of basis or subbasis elements are open.

Know how to: Check if a collection of subsets of X is a subbasis, a basis, or a topology.

3. Constructions (Sec. 16, 15, 19, 22):
Vocabulary: *Subspace topology, open in Y, product topology, projections, box topology, quotient (identification) topology, quotient map, retraction.*
Theorems:
- ∘ 16.1: Given $Y\subseteq(X,\mathcal{T}(\mathcal{B}))$, $\mathcal{B}_{subsp}=\{B\cap Y\mid B\in\mathcal{B}\}$.
- ∘ E16.1, 16.2: A subspace of a subspace is a subspace.
- ∘ 19.1: $\mathcal{B}_{prod}=\{\prod B_\alpha\mid B_\alpha\in\mathcal{T}_\alpha,\ B_\alpha=X_\alpha\ \forall$ but finitely many $\alpha\}$
- ∘ 15.1, 19.2: Given $(X_\alpha,\mathcal{T}(\mathcal{B}_\alpha))$, $\mathcal{B}'_{prod}=\{\prod B_\alpha\mid B_\alpha\in\mathcal{B}_\alpha\cup\{X_\alpha\},\ B_\alpha=X_\alpha\ \forall$ but finitely many $\alpha\}$.
- · 16.3, 19.3=E19.2: A product of subspaces is a subspace of the product.
- • (Thm I) 22.2, 22.3: Given a space X with equivalence relation $\sim$ and quotient $p:X\to X/\sim$ and a continuous function $g:X\to Z$ with $p(a)=p(b)\Rightarrow g(a)=g(b)\ \forall\, a,b\in X$, then $\exists$ continuous $f:X/\sim\to Z$ with $g=f\circ p$. Moreover, (a) if $g(a)=g(b)\Rightarrow p(a)=p(b)\ \forall\, a,b\in X$ then f is injective; (b) if g is onto then f is onto; (c) if (a) holds and g is a quotient map then f is a homeomorphism.
- • Thm Q: If $g:X\to Z$ is continuous, onto, and open, then g is a quotient map.
- • E22.2b: A retraction is a quotient map.

Know how to: Prove that a quotient space is homeomorphic to another (familiar) space.

4. Continuity (Sec. 18, 19, 20, 21, 22):
Vocabulary: *continuous, open, homeomorphism, homeomorphic* $\cong$, *homeomorphism invariant, imbedding.*
Theorems:

• 18.1: Tfae: $f : X \to Y$ is continuous; preimages of closed sets are closed; and images of closures are contained in closures of images.
• 18.2, 18.3: Constants, inclusions, composition, restriction, extension, and (local) pasting of continuous functions are continuous.
• 18.4, 19.6: Projections are continuous.
• 18.4, 19.6: $f = (f_\alpha)_{\alpha \in J} : A \to \prod X_\alpha$ is continuous iff each $f_\alpha : A \to X_\alpha$ is continuous.
• 21.5, 21.6: Given continuous $f, g : X \to \mathbb{R}$ such that $\mathbb{R}$ has the Euclidean topology, then $f + g$, $f - g$, $f \cdot g$, and f/g (if g is never 0) are continuous. If continuous $\{f_n\}$ converge uniformly to f, then f is continuous. (Cosine, sine : $\mathbb{R} \to \mathbb{R}$ are continuous.)
∘ PS3.1, PS5.2: Composition, restriction (range), projections, and products of open maps are open.
∘ PS5.1: Compositions and inverses of homeomorphisms are homeomorphisms.

Know how to: Check whether a function is continuous.

5. Closed sets and boundaries (Sec. 17):
Vocabulary: *Closed, closed in Y, interior, closure, limit point, boundary.*
Theorems:

∘ 17.1: $\emptyset$, X, and finite unions and arbitrary intersections of closed sets are closed.
· 17.2: A closed in Y iff $A = C \cap Y$ for some C closed in X.
∘ Prop.: C is closed iff $C = \overline{C}$. U is open iff $U = \text{Int } U$.
· 17.4: Closure in subspace equals intersection of closure with subspace.
· 19.5, E17.9: The product of the closures equals the closure of the product.
• 17.5: $x \in \overline{A}$ iff every open U in X that contains x intersects A.
∘ 17.6: $\overline{A} = A \cup A'$.

Know how to: Find the closure and boundary of a subset of a topological space.

6. Homeomorphism invariants (Sec. 17):
Vocabulary: *Hausdorff.*
Theorems:

∘ Thm.: Hausdorff is a homeomorphism invariant property.
∘ E17.11, E17.12, 19.4=E19.3; E22.1: Hausdorff is preserved by subspaces and products but not quotients.
· 17.8: Finite sets are closed in Hausdorff spaces.

Know how to: Check whether or not a space is Hausdorff, and use homeomorphism invariants to prove two spaces are not homeomorphic.

Math 871 Fall 2013 Midterm Review Exercises

Problem sets:
1.2cgkmpq, 2.1, 2.2, 6.2, 7.1, 7.5aei, 13.3, 13.4   PS2.1, 18.3, 18.5, PS2.2, 13.8
16.1, 16.3, PS3.1, 19.2, 19.10, 18.4, PS3.2
PS4.1, 17.3, PS4.2, 18.8 ($Y = \mathbb{R}$), 17.8ab, 17.19ab, 17.20cf, 18.2
3.4, 22.1, 22.2, PS5.1, PS5.2, PS5.3, PS5.4   17.11, 17.12, 19.3

See the course web site at www.unl.edu/~smh/871 for the PS problems that are not in the Munkres text.

Math 871 Fall 2013 Exam 2 Review

Notes:
(1) The second exam is comprehensive; see the review sheet for Exam 1 for a review of the first half of the course.
(2) You need to know complete definitions of the italicized words under Vocabulary below, along with the complete statements of the theorems paraphrased below which have a • next to their statements. You will need to be able to use all other definitions, along with the theorems paraphrased below that are preceded by a ∘. If a theorem was stated in the exercises, its number is preceded by an E. Use of the text or your notes will not be allowed on this exam.

1. Homeomorphism invariants (Sec. 20, 21, 23, 24, 25, 26, 27, 30, 31, 32, 34):
Vocabulary: *metrizable*, *connected*, *path*, *path connected*, *connected component*, *path component*, *open covering*, *subcovering*, *compact*, *second countable*, T_0, T_1,T_2 = *Hausdorff*, T_3 = *regular*, T_4 = *normal.*
Theorems:

- ∘ Thm.: Metrizable, connected, path connected, numbers of connected and path components, compact, second countable, and T_i for $0 \leq i \leq 4$ are homeomorphism invariant properties.
- ∘ 23.6, (E23.10), 23.5, E24.8; 26.7, (37.3), 26.5, 26.2: Connected, path connected, and compact are preserved by products and quotients but not subspaces.
- ∘ E21.1, E21.3, E22.1; 30.2: Metrizable and second countable are preserved by subspaces and countable products; metrizable is not preserved by quotients.
- ∘ 31.2: Hausdorff and regular are preserved by subspaces and products.
- • Prop.: Metrizable spaces are Hausdorff.
- · Prop.: U is open w.r.t. metric d iff $\forall y \in U \exists \epsilon > 0$ with $B_d(y, \epsilon) \subseteq U$.
- ∘ IVT: X connected, $f{:}X \to \mathbb{R}$ continuous, and $f(c) < r < f(d) \Rightarrow \exists\, x \in X$ with $f(x) = r$.
- · X is connected iff the only clopen subsets are $\emptyset$ and X.
- · 23.2: If $Y \subseteq X$ is connected and A, B disconnect X, either $Y \subseteq A$ or $Y \subseteq B$.
- ∘ 23.3, E24.8: If X_α are (path) connected and $\cap X_\alpha \neq \emptyset$ then $\cup X_\alpha$ is (path) connected.
- • 23.5, E24.8: A continuous image of a (path) connected space is (path) connected.
- ∘ 24.2: $Y \subseteq \mathbb{R}$ (Euclidean top.) is (path) connected iff Y is an interval, ray, or $\mathbb{R}$.
- • Prop: A path connected space is connected; converse not true in general.
- ∘ 25.5: Each connected component is a disjoint union of path components.
- • EVT: X is compact, $f{:}X \to \mathbb{R}$ continuous $\Rightarrow \exists\, a, b \in X$ such that $\forall\, x \in X$, $f(a) \leq f(x) \leq f(b)$.
- • 26.1: $Y \subseteq X$ is compact iff every covering of Y by open sets of X has finite subcovering.
- • 26.2: Every closed subset of a compact space is compact.
- • 26.3: Every compact subspace of a Hausdorff space is closed.
- · 26.4: X is T_2, $Y \subseteq X$ is compact, $x \notin Y \Rightarrow \exists$ disjoint open U, V with $x \in U$, $Y \subseteq V$.
- • 26.5: The continuous image of a compact space is compact.
- • 26.6 VUT: A continuous bijection from compact to Hausdorff is a homeomorphism.
- ∘ CHI: If $g{:}X \to Y$ is a continuous surjection, $x_1 \sim x_2$ iff $g(x_1) = g(x_2)$, X is compact, and Y is T_2, then the quotient $X/{\sim} \cong Y$.
- ∘ 26.8 Tube Lemma: If $x_0 \in X$ and N is an open set in $X \times Y$ containing $\{x_0\} \times Y$, then there is a set W open in X with $\{x_0\} \times Y \subseteq W \times Y \subseteq N$.
- ∘ 27.3: Heine-Borel: $Y \subseteq \mathbb{R}^n$ is compact iff Y is closed and bounded.
- · 27.5: Lebesgue Number Lemma: X compact metrizable, $\mathcal{C}$ open covering $\Rightarrow \exists\, \delta > 0$ such that for each $A \subseteq X$ with $diam(A) < \delta$, $\exists\, C \in \mathcal{C}$ with $A \subseteq C$.

- E30.4: Compact metrizable $\Rightarrow$ second countable. (Metrizable $\not\Rightarrow$ second countable.)
- Lem: $T_4 \Rightarrow T_3 \Rightarrow T_2 \Rightarrow T_1 \Rightarrow T_0$. For all i, $\not\Leftarrow$.
- Lem: T_1 iff all one-points sets are closed.
- 31.1: Regular iff T_1 and $\forall x \in X$ and open U with $x \in U$, $\exists$ open V with $x \in V \subseteq \overline{V} \subseteq U$. Normal iff T_1 and $\forall$ closed A and open U with $A \subseteq U$, $\exists$ open V with $A \subseteq V \subseteq \overline{V} \subseteq U$.
- 32.1: Regular and second countable $\Rightarrow$ normal.
- 32.2: Metrizable $\Rightarrow$ normal.
- 32.3: Compact and Hausdorff $\Rightarrow$ normal.
- 33.1: Let X be T_1. TFAE: (1) X is normal. (2) $\forall$ closed disjoint $A, B \subseteq X$ there is a continuous function $f{:}X \to [0,1]$ with $f(A)=0$ and $f(B)=1$.
- 34.1 (UMT): Regular and second countable $\Rightarrow$ metrizable.
- Cor: Let X be compact. TFAE: (1) X is metrizable. (2) X is T_2 and second countable.

Know how to:

- Check whether or not a space is connected, path connected, compact, second countable, T_0, T_1, Hausdorff, regular, normal and/or metrizable. Also find the connected components and path components of a space.
- Use homeomorphism invariants to prove two spaces are not homeomorphic.

2. Homotopy (H. p.1-4, 21-28; M. Sec. 51, 52, 58):

Vocabulary: I, 1_X, *retract(ion)*, *deformation retract(ion)*, *mapping cylinder*, *mapping torus*, *homotopy*, *homotopic*, *homotopy rel subspace*, *homotopy equivalence*, $\simeq$, *contractible*, nulhomotopic, *loop*, *path homotop(ic)(y)*, $\simeq_p$, straight-line homotopy, *fundamental group* $\pi_1(X, x_0)$, basepoint, $f \cdot g$, $[f]$, $[f]\cdot[g]$, c_{x_0}, *reverse* $\overline{f}$, *basepoint change homomorphism* β_f, *simply connected = 1-connected*, *induced homomorphism* h_*.

Theorems:

- PS10.2: $f{:}X \to Y$ continuous $\Rightarrow$ $Y \cong$ a deformation retract of the mapping cylinder X_f.
- Thm: $X \simeq Y$ iff $X \approx Y$, where $\approx$ is the smallest equivalence relation on spaces such that any deformation retract Y of X has $Y \approx X$.
- X is contractible iff 1_X is nullhomotopic.
- HE0.3,H1.2: $\simeq$ and $\simeq_p$ are equivalence relations on maps; $\simeq$ is an equiv. rel. on spaces.
- H1.3: $\pi_1(X, x_0)$ is a group.
- H1.5: For a path f from x_0 to x_1, $\beta_f : \pi_1(X, x_0) \to \pi_1(X, x_1)$ is an isomorphism.
- Cor: X path connected $\Rightarrow$ $\pi_1(X)$ is independent of base point up to isomorphism.
- Lem: h is continuous $\Rightarrow$ $h_*{:}\pi_1(X, x_0) \to \pi_1(Y, y_0)$ is a well-def. group homomorphism.
- Lem: $(k \circ h)_* = k_* \circ h_*$ and $(id_{(X,x_0)})_* = id_{\pi_1(X,x_0)}$.
- H1.18: X, Y path connected and homotopy equivalent $\Rightarrow$ $\pi_1(X) \cong \pi_1(Y)$.

Know how to:

- Build homotopies!

Math 871 Fall 2013 Exam 2 Review Exercises

Problem sets: See also Exam 1 Review sheet
20.3a, 21.1, PS7.1, PS7.2, PS7.3, 23.11, 24.3, 24.8
26.3, PS8.1, PS8.2, 26.5, 26.9, 30.4, 30.12 (2nd ctbl), 31.2, 31.5, 32.1, 32.2 PS9.1
H p.18 #2, PS10.1, PS10.2, H p.18 #3, PS10.3, H p.18 #9=M58.6, PS10.4, PS10.5
H p.38 #1, H p.38 #3, PS11.1

See the course web site at the math 871 link from www.unl.edu/~smh/871 for the PS problems that are not in the Munkres or Hatcher texts, and for the Exam 1 Review.