Math 872 - Algebraic Topology - Spring 2014
(Active) Table of Contents

Chapter 0: Overview and review

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    - Classification problem
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      - Groups - π_1 and H_n groups
    - Inputting spaces into computers
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    - **P1.18** Thm: If X and Y are path-connected and are homotopy equivalent, then $\pi_1(X) \cong \pi_1(Y)$. *
    - Thm: If X and Y are **path-connected** and are homotopy equivalent, then X has [abelian, finite respectively] fundamental group iff Y has [abelian, finite respectively] fundamental group.

- *Section B: Review from Math 871: Homotopy and fundamental groups*
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Chapter 6: $\pi_1(S^1)$

- *Section A: Proof*

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- **T1.7** Thm: $\pi_1(S^1) \cong \mathbf{Z}$ **
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- *Section B: Applications*

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  - **T1.10** Borsuk-Ulam Thm: If $f{:}S^2 \to \mathbf{R}^2$ is continuous, then there are antipodal points p,-p in S^2 with f(p)=f(-p).
  - **E1.1.9** Ham Sandwich Thm: Given three rectangular boxes (or more generally measurable subsets) in $\mathbf{R}^3$, there is a plane that bisects each (into regions of equal volume) simultaneously.
  - **E1.1.5** $\pi_1(X,x_0) = 1$ iff every continuous map $S^1 \to X$ extends to a continuous map $D^2 \to X$. *
  - Prop: No relationship between π_1 and **subgroup**, quotient constructions *
  - **P1.12** Thm: If X and Y are PC then $\pi_1(X \times Y)$ is isomorphic to $\pi_1(X) \times \pi_1(Y)$. **

Chapter 7: Presenting groups

- *Section A: Review and Presentations*

  - Review of normal subgroups, cosets, and quotients
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  - Definitions and examples of **presentations of groups**
  - Homomorphism Building Theorems (HBT's) **
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- *Section B: Building new groups from old*

  - Three views of direct products: Sets with multiplication, presentations, and the homomorphism building theorem (HBT, aka universal property)
  - Three views of **free products**: Sets with multiplication, presentations and the Homomorphism Building Theorem, and examples **
  - **Amalgamated products**, examples

Chapter 8: The Seifert - Van Kampen Theorem

- *Section A: Statement and first examples*

  - **T1.20** SVK Thm: If X is a union of open path-connected subspaces A_α (for indices α in an index set J) all containing the basepoint x_0, that satisfy the properties that every pairwise and triple intersection $A_\alpha \cap A_\beta$ and $A_\alpha \cap A_\beta \cap A_\gamma$ is also path-connected, then $\pi_1(X)$ is isomorphic to the group $*_{\alpha \in J}\, \pi_1(A_\alpha)/N$, where N is the normal subgroup generated by all elements of the form $i_{A_\beta A_\gamma *}([w])\, i_{A_\gamma A_\beta *}([w])^{-1}$ with $[w] \in \pi_1(A_\beta \cap A_\gamma)$. **

- Computing fundamental groups of graphs
- **Abelianization** of a group
- Applications to the homotopy equivalence problem
- **Finitely presented group** examples

- *Section B: Classification of surfaces*

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- *Section C: Proof of SVK*

  - Building the homomorphism with the HBT
  - Using the Lebesgue Number Lemma and the "seashell method" to prove onto
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Chapter 9: Presenting spaces

- *Section A: Building new spaces from old*

  - **Wedge products**
  - Thm: If for each α the basepoint x_α of each space X_α is a deformation retract of an open neighborhood U_α of x_α in X_α, then $\pi_1(\vee_\alpha X_\alpha) \cong *_\alpha \pi_1(X_\alpha)$ is a free product. **
  - **Cones**

- *Section B: CW complexes*

  - Definitions: **CW complex, cell, *n*-skeleton, attaching (characteristic) map, dimension**
  - Examples, including putting CW structures on familiar spaces
  - Continuous Function Building Thm: If X is a CW complex, Y is a topological space, and $f{:}X \to Y$ is a function, then f is continuous iff every composition $f \circ \Phi_\alpha$ of f with an attaching map is continuous. *
  - **PS(9.B.3)** Thm: A CW complex with finitely many cells is compact.
  - **PA.3** Thm: CW complexes are Hausdorff.
  - **EA.3** Thm: A CW complex X is PC iff the 1-skeleton $X^{(1)}$ is PC. **

- *Section C: Fundamental groups of CW complexes*

  - **P1A.2** Thm 1: The fundamental group of the 1-skeleton of a CW complex is a free group, generated by loops at a basepoint that follow a path in a maximal tree T, then traverse a single edge outside T, and then another path in T back to the basepoint. **
  - **P1.26** Thm 2: For a CW complex X the inclusion $X^{(1)} \to X^{(2)}$ induces a surjection of fundamental groups whose kernel is generated by loops corresponding to the attaching maps of the 2-cells of X. **

- **PS(9.C.1)** Thm n: $\pi_1(X)$ is isomorphic to $\pi_1(X^{(2)})$. **
- Presentation complexes
- **P1.28** 2-Way Street Thm: For every group G there is a 2-dimensional CW complex X with $\pi_1(X)$ isomorphic to G. *

- *Section D: Connectednesses*

  - PC and LPC, and examples
  - **Simply-connected** and SLSC, and examples
  - **PA.4** Thm: CW complexes are LPC and SLSC.

Chapter 10: Covering spaces

- *Section A: Definitions and lifting*

  - Definition and examples of **covering spaces**
  - **P1.30** Path and path homotopy lifting theorem (PPHLT) **
  - **P1.31** Thm: If $p:Y \to X$ is a covering space then ker $p_* = 1$ and im $p_* = \{[f] |\ f$ lifts to a loop$\}$. **
  - Application/Cor: For all natural numbers n, the free group F_n is a subgroup of the free group F_2.
  - **P1.32** Lifting Correspondence Function and Thm: Let X and Y be path-connected spaces and let p: (Y,y_0) -> (X,x_0) be a covering space. The **Lifting Correspondence Function** Φ: $\pi_1(X,x_0)$ / $p_*(\pi_1(Y,y_0))$ -> $p^{-1}(\{x_0\})$ defined by $\Phi(p_*(\pi_1(Y,y_0))[l]) := m(1)$, where m is the unique lift of the path l in Y starting at y_0, is a well-defined bijection. **
  - **P1.33,P1.34** Lifting Criterion and Unique Lifting Property, and examples **
  - Cor: If p: Y -> X is a covering space and X is a CW complex, then Y is a CW complex. *

- *Section B: Group actions*

  - **Group action, covering space action, orbit, and orbit space**
  - **P1.40** Thm: If G has a covering space action on Y, then: (1) The quotient p:Y -> Y/G is a covering space. (2) If Y is PC and LPC then $p_*(\pi_1(Y,y_0))$ is normal in $\pi_1(Y/G,[y_0])$. (3) If Y is PC and LPC then G is isomorphic to $\pi_1(Y/G,[y_0])$ / $p_*(\pi_1(Y,y_0))$. **
  - Cor: If Y is simply-connected and LPC and G has a covering space action on Y, then G is isomorphic to $\pi_1(Y/G)$. **
  - Definitions of **presentation complex**, and **Cayley complex**, and examples.
  - **Hp.77** Thm: Let Y be the Cayley complex of a presentation of G. Then G has a covering space action on Y, Y/G is the presentation complex, Y is simply connected, and $\pi_1(Y/G) \cong G$. *

- *Section C: The universal covering and the Galois correspondence*

  - Existence theorems
    - **P1.36** Simply-connected Covering Thm: Let X be a PC, LPC, SLSC space. Then there is a simply-connected covering space p: Y -> X, and there is a covering space group action of $\pi_1(X)$ on Y inducing the map p. **
    - **P1.36** Existence Thm: Let X be a PC, LPC, SLSC space, and let H be a subgroup of

$\pi_1(X)$. Then there is a covering space p: Y -> X with $H=p_*(\pi_1(Y))$. Moreover, if H is a normal subgroup of $\pi_1(X)$ then there is a covering space group action of G/H on Y inducing the map p. *

- Uniqueness theorem
  - Definition of **isomorphism of pointed covering spaces**
  - Definition of **deck transformations**
  - **P1.37** Uniqueness Thm: Any two PC, LPC pointed covering spaces p_i: (X_i,x_i) -> (X,x_0) (i=1,2) of a pointed space (X,x_0) satisfy $p_{1*}(X_1,x_1) = p_{2*}(X_2,x_2)$ iff the pointed covering spaces are isomorphic.
- MegaTheorem: Let (X,x_0) be a PC, LPC, SLSC space, let $G = \pi_1(X,x_0)$, and let p: (Y,y_0) -> (X,x_0) be the simply-connected covering. Let q: (Z,z_0) -> (X,x_0) be any PC covering space. Then:
  (a) **P1.38** Galois Correspondence Thm: The maps {subgroups H of G} <-> {isomorphism classes of PC pointed coverings of (X,x_0)} defined by H -> (p': $(Y/H,[y_0])$ -> (X,x_0)) and (p'': (Y'',y''_0) -> (X,x_0)) -> im p''_* are inverse bijections. **
  (b) **Hp.68** Universal Covering Thm: There is a covering space map r: Y -> Z with the composition qr=p. **
  (c) **P1.39** Deck Transformation Thm: Deckgp(q) = N_G(H)/H, where H := $r_*(\pi_1(X,x_0))$. Moreover, H is normal in G iff Deckgp(q) (=G/H) has a covering space action on Y inducing q. *
- The correspondence table and examples
- **T1A.4** Application/Thm: Every subgroup of a free group is free.
- The "look up" and "look down" methods of constructing the covering space corresponding to a given subgroup, and more examples

Chapter 11: Simplicial homology

- *Section A: Overview of homology*
  - Higher homotopy groups and their weaknesses
  - Strengths and weaknesses of simplicial, CW, and singular homologies
- *Section B: Δ-complexes*
  - **Standard simplices** and **faces**
  - **Δ-complex** definition and examples
- *Section C: Simplicial homology*
  - Definition of **Simplicial n-chains** and **boundary maps ∂_n**
  - **L2.1** Lem: ∂_{n-1} o ∂_n
  - Definition of **simplicial homology groups H_n^{simpl}**
  - Examples
  - Linear algebra over **Z**
  - Connections between ker ∂_1 and loops, im ∂_2 and disks filling in loops

Chapter 12: Singular homology

- *Section A: Definitions and induced homomorphisms*

  - **Singular chains** and definition of $\mathbf{H_n}^{\mathbf{sing}}$
  - **P2.6** Thm: If X has path components X_α, then $H_n^{sing}(X) = \oplus_\alpha H_n^{sing}(X_\alpha)$. *
  - **P2.7** Thm: If X is path-connected, then $H_0^{sing}(X) = \mathbf{Z}$. **
  - **Reduced singular homology**
  - Homological algebra excursion:
    - **Chain complex**, **cycle**, **boundary**, **homology**
    - **Chain map**, **induced homology homomorphism**
    - Chain homotopy
    - **P2.12** Thm: Chain homotopic chain maps induce the same homology homomorphism.
  - **T2.27** Thm: If X is a Δ-complex, then $H_n^{simpl}(X)$ and $H_n^{sing}(X)$ are isomorphic for all n. *
  - Inducing singular homology homomorphisms via continuous functions
  - **T2.10** Thm: Homology homomorphisms induced by continuous functions are abelian group homomorphisms that compose nicely; homotopic maps induce the same homology homomorphism.
  - **C2.11** Thm: If X and Y are homotopy equivalent, then $H_n^{sing}(X)$ and $H_n^{sing}(Y)$ are isomorphic for all n. **
  - Cor: If a topological space has two Δ-complex structures, i.e. if X and X' are homeomorphic Δ-complexes, then $H_n^{simpl}(X)$ is isomorphic to $H_n^{simpl}(X')$ for all n.
  - Examples/applications

- *Section B: Mayer-Vietoris Theorem*

  - Statement and first examples
    - Homological algebra: **Exact sequence**
    - Lem: Let a: $A \to B$ be an abelian group homomorphism. (1) $0 \to A \to B$ is exact iff a is one-to-one. (2) $A \to B \to 0$ is exact iff a is onto. *
    - **H p.149** MV Thm: Suppose that X is a topological space with subspaces A,B such that X = Int(A) $\cup$ Int(B). Then there is an exact sequence $\ldots \to H_n(A \cap B) \to H_n(A) \oplus H_n(B) \to H_n(X) \to H_{n-1}(A \cap B) \to \ldots \to H_0(X) \to 0$, such that each homorphism $\varphi_n: H_n(A \cap B) \to H_n(A) \oplus H_n(B)$ is given by $\varphi_n = (i_{AB\,n,*}, -i_{BA\,n,*})$ and each homomorphism $\psi_n: H_n(A) \oplus H_n(B) \to H_n(X)$ is given by $\psi_n = j_{A\,n,*} + j_{B\,n,*}$, where $i_{AB}: A \cap B \to A$, $i_{BA}: A \cap B \to B$, $j_A: A \to X$, and $j_B: B \to X$ are inclusion maps. **
    - Connection to the SVK theorem
    - Examples of computing with the Mayer-Vietoris Theorem
  - Proof of MV and the definition of the δ homomorphism
    - Small Chains Thm: If A,B are subspaces of X with X=Int(A) $\cup$ Int(B) then $H_n(X)$ is isomorphic to $H_n(A+B)$.
    - Picture for using the Lebesgue Number Lemma to prove the Small Chains Thm.
    - **T2.16** Snake Lemma: A short exact sequence of chain complexes induces a long exact sequence on homology. *

- *Section C: Simplifying H_n with subspaces*

- Definition and long exact sequence for **relative homology**
- **T2.13** Crushing Good Subspaces Thm: If A is a nonempty closed subspace of X that is a deformation retract of an open neighborhood of A in X, then $H_n(X,A) \cong H_n(X/A)$ for $n > 0$. **
- **T2.20** Excision Thm: (a) If $Z \subseteq A \subseteq X$ and $Cl(Z) \subseteq Int(A)$, then $H_n(X,A) \cong H_n(X - Z, A - Z)$. (b) If $A,B \subseteq X$ and $X = Int(A) \cup Int(B)$, then $H_n(X,A) \cong H_n(B, A \cap B)$. **
- Examples

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