Math 872 - Section 001 - Spring 2014 - Problem sets

- Problem set 1:
  - **(6.A.1)** Hatcher p. 38 # 5
    (Hint: You may wish to use the following lemma in your proofs:
    Lemma *: If $g : X \to Y$ is a continuous surjection, X is compact, and Y is Hausdorff, then g is a quotient map.)
  - **(6.A.2)** For each integer n, let ω_n be the path in S^1 defined by $\omega_n(s) := (\cos(2 \pi n s) , \sin(2 \pi n s))$ for all $s \in I$. Define a homotopy $H : I \times I \to S^1$ by
    $H(s,t) := (\cos(2 \pi m (s/C)) , \sin(2 \pi m (s/C)))$ for $s \in [0,C]$ and
    $H(s,t) := (\cos(2 \pi n ((s - C)/(1 - C))) , \sin(2 \pi n ((s - C)/(1 - C))))$ for $s \in [C,1]$
    where $C := (1-t)(1/2)+tm/(m+n)$. (a) Prove that H is a path homotopy from $\omega_m * \omega_n$ to ω_{m+n}. (b) For some values of m and n, the statement in part (a) is false(!); determine what these values are, and find a homotopy between $\omega_m * \omega_n$ to ω_{m+n} for those m and n.
  - **(6.A.3)** Using the map $p : \mathbf{R} \times \mathbf{R} \to S^1 \times S^1$ be defined by $p(x,y) := ((\cos(2 \pi x) , \sin(2 \pi x)) , (\cos(2 \pi y) , \sin(2 \pi y)))$:
    (a) Let f be the loop in $S^1 \times S^1$ based at ((1,0),(1,0)) defined by $f(s) := ((\cos(2 \pi s) , \sin(2 \pi s)) , (\cos(4 \pi s) , \sin(4 \pi s)))$. Find a path $g: (I,0) \to (\mathbf{R} \times \mathbf{R},(0,0))$ satisfying $p \circ g = f$. Sketch the image of g in $\mathbf{R} \times \mathbf{R}$ and the image of f in $S^1 \times S^1$ (viewing $S^1 \times S^1$ as the surface of a doughnut).
    (b) Generalize the proof that $\pi_1(S^1)=\mathbf{Z}$ (and the proofs of the path and homotopy lifting theorems) to prove that $\pi_1(S^1 \times S^1) = \mathbf{Z} \times \mathbf{Z}$.
  - **(6.A.4)** Consider the map $h: S^1 \to S^1$ given by $h(z)=1/(z^n)$ (where $z \in S^1$ is considered as a complex number). What group homomorphism : $\mathbf{Z} \to \mathbf{Z}$ is induced by h? (Prove your answer.)
  - **(6.A.5)** Show that every homomorphism $\varphi : \mathbf{Z} \to \pi_1(X,x_0)$ can be realized as the induced homomorphism $\varphi = h_*$ of a continuous map $h : (S^1, (1,0)) \to (X,x_0)$.
    (Hint: Lemma * above can be useful for the proof here, too.)
  - **(6.B.1)** Hatcher p. 39 # 16abf
    (Hints for f: The boundary of X is the set of points that have an open neighborhood homeomorphic to the half disc $\{(x,y) \in \mathbf{R}^2 \mid x^2 + y^2 < 1 \text{ and } x \geq 0\}$. Let x_0 be a basepoint of X on the boundary circle. Let f be the loop in X at x_0 that goes around the center circle of X once, and let g be the loop at x_0 that goes around the boundary cricle of X once. How are f and g related? Where does the map induced by a retraction $r:X \to A$ send [f] and [g]?)
  - **(6.B.2)** Hatcher p. 39 # 10 (Remember Hatcher means *path* homotopy here. Prove your answer.)
  - **(6.B.3)** Hatcher p. 38 # 8

  Problems due 1/30/14 for grading: 6.A.1 (just the equivalence of b and c), 6.A.5, 6.B.2, 6.B.3

- Problem for presentation in class:
  - **(PP.1)** Let A_1, A_2, and A_3 be solid rectangular blocks in $\mathbf{R}^3$. For each q in S^2 and t in $\mathbf{R}$, let $P_{q,t}$ be the plane in $\mathbf{R}^3$ containing the point tq and perpendicular to the vector pointing from (0,0,0) to q. Let $H_{q,t}$ be the connected component of $\mathbf{R}^3 - P_{q,t}$ containing the point (t+1)q and let $J_{q,t}$ be the other connected component.

(a) Draw pictures of $P_{q,t}$ and $H_{q,t}$. Using pictures and multivariate calculus (Math 208), discuss how to compute the volume $vol(A_i \cap H_{q,t})$.
(b) Discuss why the function $S^2 \times \mathbf{R} \to \mathbf{R}$ given by $(q,t) \to vol(A_3 \cap H_{q,t})$ is continuous. Prove that for each q in S^2 there is a real number t_q such that $vol(A_3 \cap H_{q,t_q}) = vol(A_3 \cap J_{q,t_q})$. Discuss why t_q is unique.
(c) For i = 1,2 define the functions $f_i : S^2 \to \mathbf{R}$ by $f_i(q) := vol(A_i \cap H_{q,t_q})$. Discuss why these functions are continuous.
(d) Use the Borsuk-Ulam Theorem to prove that there is a plane in $\mathbf{R}^3$ that simultaneously divides each block A_i into two pieces of equal volume.
(*Note:* In parts that say "discuss" you don't need to give a full proof; a discussion with pictures will do.)

- Problem set 2:
  - **(7.A.1)** In each part of this problem, a group G is given by a presentation, together with a familiar group H. Prove that the groups G and H are isomorphic.
    (a) $G = < a \mid a^3 >$; H is the cyclic group of order 3.
    (b) $G = < a,b,c \mid a^2, b^2, c^2, (ab)^3, (ac)^2, (bc)^3 >$; H is the symmetric group permuting 4 objects.
  - **(7.B.1)** Let $B := < a,b \mid aba = bab >$ (this is the *braid group on 3 strands*). Let $H := < x \mid >$, $K := < y \mid >$, and $L := < z \mid >$ with group homomorphisms c : L -> H defined by $c(z) := x^2$ and d : L -> K defined by $d(z) := y^3$. Let G be the free product of H and K amalgamated along L via the maps c and d; that is, $G = H *_L K$. Prove that the groups B and G are isomorphic.
  - **(7.B.2)** Hatcher p. 52 # 1. (Hint: Use the reduced sequence view.)
  - **(8.A.1)** Use the methods of section 8.A to compute the fundamental group of the Mobius band.
  - **(8.A.2)** Hatcher p. 53 # 4
  - **(8.A.3)** (a) Explain how the 2-sphere S^2 is (homeomorphic to) a quotient of the square I x I with a choice of identifications among the edges on the boundary of the square (i.e. identifying sets of edges with arrows). (Show your map by pictures only.)
    (b) Let X be the quotient of S^2 obtained by identifying the north and south poles to a single point. Using part (a) and the Seifert-van Kampen Theorem, compute $\pi_1(X)$. (This part does need proof details.)
  - **(8.B.1)** Let X be an octagon in $\mathbf{R}^2$. Define an equivalence relation on X corresponding to labeling the 8 edges in the boundary of X in a counterclockwise fashion in order by: counterclockwise a, counterclockwise b, clockwise a, clockwise b, counterclockwise c, counterclockwise d, clockwise c, clockwise d. Let M be the corresponding quotient space.
    (a) Recall the concrete version of M you built in Math 871 out of paper or cloth (or any other 2-dimensional flexible material): M is homeomorphic to the frosting on a doughnut with g holes; what is g?
    (b) Compute $\pi_1(M)$ and prove your answer.
    (c) What familiar group is isomorphic to the abelianization $\pi_1(M)^{ab}$? (Prove your answer.)
    (d) Show that M is not homotopy equivalent to the torus T^2, 2-sphere S^2, projective plane P^2, or Klein bottle K^2.

Problems due 2/13/14 for grading: 7.B.1, 7.B.2 (just the proof that G*H has trivial center), 8.A.3

- Problem set 3:
  - **(8.B.2)** Use the proof of the Classification of Surfaces theorem to determine which of the spaces listed in that classification is homeomorphic to the Klein bottle.
  - **(8.C.1)** Find an example of a topological space X that is a union of three open path-connected subspaces A_i (for i=1,2,3) such that each pairwise intersection $A_i \cap A_j$ is path-connected, but $\pi_1(X)$ is not isomorphic to the group $*_{i=1}^{3} \pi_1(A_i)/N$, where N is the normal subgroup generated by all elements of the form $i_{A_i A_j *}([w])\, i_{A_j A_i *}([w])^{-1}$ with $[w] \in \pi_1(A_i \cap A_j)$.
  - **(9.A.1)** Let X be $\mathbf{R}^3$ - ({ (x,0,0) | x $\in$ **R** } $\cup$ { (1,1,1), (2,3,4) }) with the Euclidean subspace topology. Then X deformation retracts to a subspace that is homeomorphic to a wedge product of m copies of S^1 and n copies of S^2. What are m and n? Use your answer to compute $\pi_1(X)$. (Formal proof not needed for this problem.)
  - **(9.B.1)** For a CW complex X, show that if the 1-skeleton $X^{(1)}$ is path-connected, then so is X.
  - **(9.B.2)** Hatcher p. 19 # 17
  - **(9.B.3)** Show that every CW complex with finitely many cells is compact.
  - **(9.B.4)** Let X be a CW complex with finitely many cells, and for each natural number n, let j_n be the number of n-cells of X (that is, j_n = |J_n|, where J_n is the index set for the n-cells). The **Euler characteristic** for X is defined to be $\chi(X) := \Sigma_{n \in \mathbf{N}} (-1)^n$j_n.
    (a) Compute the Euler characteristic of all of the compact connected surfaces.
    (b) Find a CW structure for the cone CX of X, and determine the Euler characteristic of CX.
  - **(9.C.1)** Show that if the map f : S^{n-1} -> X is used to attach an n-cell, with n > 2, to X to form a space Y, then the inclusion X -> Y induces an isomorphism from $\pi_1(X)$ to $\pi_1(Y)$. Show that the same is true if we attach any (finite or infinite) collection of cells of dimension > 2.
  - **(9.C.2)** Hatcher p. 53 # 8
  - **(9.C.3)** Let X be the topological space obtained by taking a quotient of a Euclidean triangle, identifying the three vertices. Find a CW complex structure for X and use this structure to compute the fundamental group for X.

  Problems due 2/27/14 for grading: 8.C.1, 9.A.1, 9.B.1, 9.C.2

- Problem for presentation in class:
  - **(PP.2)** Let X be the space obtained from the solid cube I x I x I by gluing opposite square faces to one another with a 90-degree righthand twist (e.g., glue I x I x {0} to I x I x {1} by identifying (x,y,0) with (y,1-x,1)). Describe a CW structure for X and compute a presentation for $\pi_1(X)$.

- Problem set 4:
  - **(10.A.1)** Hatcher p. 79 # 1
  - **(10.A.2)** Hatcher p. 79 # 3
  - **(10.A.3)** (a) Show (by pictures) a 2-sheeted covering space of the Klein bottle by the torus. (That is, p:T^2 -> K^2.)
    (b) Show (by pictures) a simply connected covering space of the Klein bottle.
    (c) Using your answer in either (a) or (b) and the path lifting theorem, show that the fundamental group of the Klein bottle is not abelian.
  - **(10.A.4)** Lifting Criterion/Unique Lifting Property Proof Deconstruction: The following questions refer to the proofs in Hatcher on p. 62-3; LC para. 1 line 1 starts with ``The 'only if' statement is '',

LC para. 2 line 3 starts with ``path-connected open neighborhood'', ULP line 4 starts with ``continuity of \tilde f_1'', etc.

    - (a) LC para 1 line 3: What theorem is used in the definition of the function \tilde f: Y -> \tilde X?
    - (b) LC para 1 lines 6-7: What theorem is Hatcher applying in order to know that a loop h_1 exists such that h_1 is path homotopic to h_0 and the lift \tilde h_1 at \tilde x_0 is a *loop*?
    - (c) LC para 1: Where is the hypothesis that im $f_* \subseteq$ im p_* used in the proof that \tilde f is well-defined?
    - (d) LC para 2: In order to prove that \tilde f is continuous, Hatcher is proving another condition that is equivalent to the definition of continuous. State this condition, and give a reference to the theorem in Munkres that shows the equivalence. Also, Hatcher doesn't show the condition for all open neighborhoods of f(y), he only shows it for evenly covered ones; write the missing sentences that show the condition holds for all open neighborhoods of f(y).
    - (e) LC paras 1-2: Exactly where are the hypotheses that Y is PC and LPC, respectively, used in this proof?
    - (f) ULP lines 4-5: Hatcher is again applying the theorem from Munkres you found in part (d) here. Write a couple of extra lines explaining more clearly why an open set N exists with the properties listed in these lines.
    - (g) ULP: Exactly where is the hypothesis that Y is connected used in this proof?
  - **(10.B.1)** Let Y be a path-connected, locally path-connected space with a covering space action by a group G, let $y_0 \in Y$, and let p: Y -> Y/G be the corresponding quotient map. Prove that $p_*(\pi_1(Y,y_0))$ is a normal subgroup of $\pi_1(Y/G,[y_0])$.

Exam 1 3/11/14

- Problem for presentation in class:
  - **(PP.3)** Let G be the group presented by $G = < a,b \mid a^2=1, b^3=1 >$. Let H be the subgroup of G generated by a. Let $h : G \rightarrow \mathbf{Z}/2\mathbf{Z} = < c \mid c^2 = 1 >$ be the homomorphism built using the HBT that satisfies h(a) = c and h(b) = 1, and let K be the kernel of h.
    (a) Explain how to view G as a free product of two finite groups, and use this to write out all of the elements of G as reduced sequences. Describe/draw the presentation complex X and Cayley complex Y for this presentation of G.
    (b) Prove that H is isomorphic to a familiar group. Draw/describe the quotient Y/H of Y by the action of H.
    (c) Determine which reduced sequences in the free product group G do (or do not) lie in the subgroup K. Use this information to draw/describe the quotient Y/K of Y by the action of K. Find a presentation for the fundamental group $\pi_1(Y/K)$ of the CW complex Y/K.

- Problem set 5:
  - **(10.B.2)** Let G be the group presented by $< a,b \mid b^2 = 1 >$.
    (a) Draw the presentation complex for this presentation, and write this complex as a wedge sum of two familiar spaces.
    (b) Draw the Cayley graph and the Cayley complex for this presentation.
    (c) Draw a simply connected covering space of the wedge sum of a circle S^1 and a 2-sphere S^2.

How does this differ from the Cayley complex in part (b)?

- **(10.B.3)** Let Y be a simply connected and LPC space. Let G and H be subgroups of the group Homeo(Y) of all homeomorphisms of Y. Suppose that G and H are conjugate in Homeo(Y), and that both G and H give covering space actions on Y. Show that Y/G and Y/H are homeomorphic spaces.
- **(10.C.1)** Hatcher p. 80 # 12
- **(10.C.2)** For the symmetric group permuting 3 objects, $S_3 = < a,b \mid a^2 = b^2 = (ab)^3 = 1 >$: Find the presentation complex, the Cayley complex, and all connected covering spaces of the presentation complex. Describe all of the cells and attaching maps of these CW complexes, in addition to drawing pictures.
- **(10.C.3)** Hatcher p. 82 # 31
- **(10.C.4)** Let G be a finitely generated group and n a natural number.
  (a) Show that if H is an index n subgroup of G then H is finitely generated. Moreover, show that if G is finitely presented then so is H.
  (b) Show that there are only a finite number of subgroups of G of index n.
  (*Hint:* In both parts, consider coverings of the presentation complex for G. In part (b), consider the case that G is a free group first.)

Problems due 4/3/14 for grading: 10.B.3, 10.C.1

- Problem set 6:
  - **(11.B.1)** Hatcher p. 131 # 2
  - **(11.B.2)** Construct a Δ-complex structure for the connected sum of two tori. (Give an explanation with pictures, formal proof not needed. Be sure to label all simplices of all dimensions.)
  - **(11.C.1)** Hatcher p. 131 # 4 (Complete proof with all maps written out, etc., is needed here.)
  - **(11.C.2)** Compute the simplicial homology groups of the Klein bottle, torus, projective plane, and Mobius band.
  - **(11.C.3)** Let X be the standard 3-simplex Δ^3 with the "standard" Δ-complex structure (that is, with four 0-simplices, six 1-simplices, four 2-simplices, and one 3-simplex). Compute the simplicial homology groups of X, and compare them to the simplicial homology groups computed in class for the 2-sphere S^2.
  - **(11.C.4)** Hatcher p. 131 # 9

  Problems due 4/17/14 for grading: 11.B.2, 11.C.1

- Problems for presentation in class:
  - **(PP.4)** Let X be the topological space built in problem PP.2. Describe a Δ-complex structure for X, and compute the homology groups $H^{simpl}{}_n(X)$ for all n. (Hint: Use four prisms.)
  - **(PP.5)** Let Y be any finite path-connected graph. Show how to compute the simplicial homology groups of Y, and write these groups in terms of the Euler characteristic of Y (see problem (9.B.4) for the definition).

- Problem set 7:
  - **(12.A.1)** Hatcher p. 132 # 11
  - **(12.A.2)** Hatcher p. 133 # 29
  - **(12.B.1)** Use the Mayer-Vietoris Theorem to compute the homology groups of the following spaces:

(a) The space obtained from two tori glued along a circle from problem (9.C.2)/Hatcher p. 53 # 8.
(b) The square I × I with all four boundary edges glued in a clockwise orientation.
(c) The mapping cylinder of the function f:I → I defined by f(t) = 3t for t in [0,1/3], f(t) = -3t+2 for t in [1/3,2/3], and f(t) = 3t-2 for t in [2/3,1].

- **(12.B.2)** Hatcher p. 158 # 31
- **(12.B.3)** Let X be a path-connected compact Hausdorff space. The *cone* on X is the quotient CX of the space X x I by the equivalence relation making (p,0) equivalent to (q,0) for all p,q in X, and the *suspension* of X is the quotient SX of the space X x I by the equivalence relation making (p,0) equivalent to (q,0) and (p,1) equivalent to (q,1) for all p,q in X.
(a) If X is a Δ-complex, find Δ-complex structures for CX and SX.
(b) Show that the reduced homology groups of CX satisfy \tilde H_i(CX) = 0 for all i.
(c) Show that the reduced homology groups of SX satisfy \tilde H_i(SX) = \tilde H_{i-1}(X) for all i.

Problems due 4/24/14 for grading: 12.A.2, 12.B.1(b)
Exam 2 due 5/4/14

S. Hermiller.