PROBLEMS IN ALGEBRAIC TOPOLOGY

LAURENTIU MAXIM

1. Manifolds and Poincaré Duality

(1) Show that if a connected manifold M is the boundary of a compact manifold, then the Euler characteristic of M is even.
(2) Show that $\mathbb{RP}^{2n}$, $\mathbb{CP}^{2n}$, $\mathbb{HP}^{2n}$ cannot be boundaries.
(3) Show that if M^{4n} is a connected manifold which is the boundary of a compact oriented $(4n+1)$-dimensional manifold V, then the signature of M is zero.
(4) Show that $\mathbb{CP}^2 \# \mathbb{CP}^2$ cannot be the boundary of an orientable 5-manifold. However, it is the boundary of a non-orientable manifold.
(5) Show that if M^n is connected, non-compact manifold, then $H_i(M;R) = 0$ for $i \geq n$.
(6) The Euler characteristic of a closed manifold of odd dimension is zero.
(7) True or False: Any orientable manifold is a 2-fold covering of a non-orientable manifold.
(8) Show that the Euler characteristic of a closed, oriented, $(4n+2)$-dimensional manifold is even.
(9) Let M be a closed, oriented $4n$-dimensional manifold. Show that the signature $\sigma(M)$ is congruent mod 2 to the Euler characteristic $\chi(M)$.
(10) If M is a compact, closed, oriented manifold of dimension n, show that the torsion subgroups of $H^i(M)$ and $H^{n-i+1}(M)$ are isomorphic.
(11) Let M be a closed, connected, n-dimensional manifold. Show that:

$$\mathrm{Tor}(H_{n-1}(M;\mathbb{Z}) = \begin{cases} 0, \text{if M is orientable} \\ \mathbb{Z}_2, \text{if M is non-orientable} \end{cases}$$

(12) Show that $\mathbb{RP}^{2n}$ cannot be embedded in S^{2n+1}.
(13) Show that $\mathbb{RP}^{2n}$, $\mathbb{CP}^{2n}$ are fixed-point free spaces.

2. Homotopy Theory

(1) Find spaces with the same homotopy (homology) groups, but having different homotopy type.
(2) Describe the cell structure on lens spaces.
(3) Show that $V_n(\mathbb{R}^\infty)$, $V_n(\mathbb{C}^\infty)$, $V_n(\mathbb{H}^\infty)$ are contractible.

3. Cohomology ring

(1) Find the cohomology ring of real/complex projective spaces and of lens spaces.
(2) Calculate $H^*(\mathbb{RP}^\infty;\mathbb{Z})$, $H^*(\mathbb{RP}^{2n};\mathbb{Z})$, $H^*(\mathbb{RP}^{2n+1};\mathbb{Z})$.
(3) Prove Borsuk-Uhlam theorem.

(4) If there exists a division algebra on $\mathbb{R}^n$, then $\mathbb{RP}^{n-1}$ is orientable, hence $n = 2^r$, for some r.
(5) Find the ring structure on $H^*(SU(n))$, $H^*(U(n))$, $H^*(V_k(\mathbb{C}^n))$.
(6) Find the cohomology ring of $\Omega\mathbb{CP}^n$, $\Omega\mathbb{RP}^n$. (Hint: use a spectral sequence argument)

4. Spectral Sequences and Applications

(1) Find the ring structure on $H^*(SU(n))$, $H^*(U(n))$, $H^*(V_k(\mathbb{C}^n))$.
(2) Prove the Suspension Theorem for homotopy groups.
(3) Calculate $H_*(\Omega S^n)$ and the ring structure on $H^*(\Omega S^n)$.
(4) Find the cohomology ring of $\Omega\mathbb{CP}^n$, $\Omega\mathbb{RP}^n$. (Hint: use a spectral sequence argument)
(5) Find the cohomology ring of Lens Spaces.
(6) Calculate $H^*(K(\mathbb{Z}, 3))$.
(7) Find $H^*(K(\mathbb{Z}, n); \mathbb{Q})$.
(8) Calculate $\pi_4(S^3)$, $\pi_5(S^3)$.
(9) Show that the p-torsion in $\pi_i(S^3)$ appears first for $i = 2p$ and it is $\mathbb{Z}_p$.
(10) Where does the 7-torsion appear first in the homotopy groups of S^n ?
(11) Prove Serre's theorem: (a) The homotopy groups of odd spheres S^n are torsion except in dimension n; (b) The homotopy groups of even spheres S^n are torsion except in dimension n and $2n-1$.
(12) Calculate the cohomology of the space of maps from $S^1 \to S^3$, and similarly for the space of maps $S^1 \to S^2$ and $S^1 \to \mathbb{CP}^n$.
(13) Find the cohomology ring of $BU(n)$, $BO(n)$.

5. Fibre Bundles

(1) Classify the S^1-bundles over S^2.
(2) Show that any vector bundle over a simply-connected base space is orientable.
(3) Show that if an oriented vector bundle has a non-zero section, then it's Euler class is zero.
(4) Construct an orientable sphere bundle with zero Euler class, but no section.

6. Characteristic Classes

(1) Calculate $w(\mathbb{RP}^n)$, $c(\mathbb{CP}^n)$, $p(\mathbb{CP}^n)$.
(2) Show that a manifold M is orientable if and only if its first Stiefel-Whitney class vanishes.
(3) Study possible immersions (embeddings) of $\mathbb{RP}^n$ into $\mathbb{R}^{n+k}$. (Hint: use S-W classes)
(4) Show that the only real projective spaces which can be parallelizable are $\mathbb{P}^1$, $\mathbb{P}^3$ and $\mathbb{P}^7$.
(5) Show that if n is a power of 2, then $\mathbb{RP}^n$ cannot be smoothly embedded in $\mathbb{R}^{2n-1}$.
(6) Show that $\mathbb{RP}^{2k+1}$ cannot be a boundary.
(7) Show that $\mathbb{CP}^4$ cannot be smoothly embedded in $\mathbb{R}^n$ with $n \leq 11$. (Hint: use Pontrjagin classes)
(8) Find the smallest k such that $\mathbb{CP}^n$ can be smoothly embedded in $\mathbb{R}^{2n+k}$.

(9) Show that $\mathbb{CP}^{2n}$ is not an oriented boundary. (Hint: use the Pontrjagin number)
(10) Find obstructions to the existence of a complex structure on an even dimensional manifold.
(11) Find the cohomology ring of $BU(n)$, $BO(n)$.
(12) Show that if M is an oriented boundary, then all its Pontrjagin numbers are zero.
(13) $\mathbb{CP}^n$ cannot be expressed non-trivially as a product of complex manifolds.