Topology Qual Workshop Day 6: Homotopy & Retractions

Warm-up Problems:

- Let $A \subseteq X$ and $r : X \to A$ a retraction. Prove that $r_* : \pi_1(X) \to \pi_1(A)$ is a surjection.
- Let $B \subseteq Y$ and let $p : Y \to B$ be a deformation retraction. Prove that $p_* : \pi_1(Y) \to \pi_1(B)$ is an isomorphism.

(1) (Michigan May '10) Let $\mathbb{R}P^2$ and T denote, in this order, the real projective plane and the torus $S^1 \times S^1$. Prove that any map

$$f : \mathbb{R}P^2 \to T,$$

is homotopic to a constant map.

Variration: Let X be a path connected, locally path connected space with $\pi_1(X)$ finite. Show that every continuous map $f : X \to S^1$ is homotopic to a constant map.

(2) (Michigan Jan '10) Let D^2 be a closed 2-disk, S^1 its boundary. Is the space $X = S^1 \times S^1$ a retract of the space $Y = D^2 \times S^1$? (i.e. does there exist a continuous map $Y \to X$ which is the identity on $X \subset Y$?)

(3) (Arizona Winter '05) Suppose $\phi : S^1 \to S^1$ and $\psi : S^1 \to S^1$ are continuous maps. Show that the compositions $\phi \circ \psi$ and $\psi \circ \phi$ are homotopic. (Hint: is this true for $f(z) = z^n$ where $n \in \mathbb{Z}$?)

(4) (Arizona Fall '05) Let $f : D^2 \to D^2$ be a homeomorphism (where D^2 is the closed unit disk in $\mathbb{R}^2$). Show that f must map boundary points $x \in \partial D^2$ to points $f(x) \in \partial D^2$ on the boundary.

(5) (Arizona Spring '05) Let $f : S^1 \vee S^1 \to T$ be a continuous map, where T is the torus. Show that there is no continuous map $g : T \to S^1 \vee S^1$ such that $f \circ g$ is the identity map on T.

(6) Prove that S^{n-1} is not a retract of B^n.

Covering Space Problems (Suggested reading: Hatcher, page 77 - 78):

- Let $G = \langle x | x^2 \rangle$. Find the presentation complex, X_G, which is a familiar space, identify it as such. Then find the universal cover of X_G, $\tilde{X}_G$.
- Let $H = \langle x | x^6 \rangle$. Find the presentation complex, X_H. Then find the universal cover of X_H, $\tilde{X}_H$.