Algebra: Suggested Questions

1. (a) If A and B are subgroups of finite index in a group G, and $[G : A]$ and $[G : B]$ are relatively prime, then $G = AB$.

   Solution: Firstly, if $[G : B]$ is finite then, for any subgroup A of G, $[A : A \cap B]$ is finite and $\leq [G : B]$ and moreover equality holds if and only if $G = AB$. To see this let $g_1B, \ldots, g_nB$ be the left cosets of B such that $g_iB \cap A \neq$ and let $a_i \in g_iB, a_i \in A$ for each i. Then, the $a_i(A \cap B)$ are the cosets of $A \cap B$ in A. Also, if $g_1B, \ldots, g_nB$ are all the cosets of B, then $[G : B] = [A : A \cap B]$ and clearly every element of $G \in AB$.

   Now, since $A \cap B$ is a subgroup of both A and B, we have that $[G : A \cap B] = [G : A][A : A \cap B]$, and $[G : A \cap B] = [G : B][B : A \cap B]$. Since $[G : A]$ and $[G : B]$ are mutually prime, this shows that $[G : A \cap B] \geq [G : A][G : B]$. But, from the previous paragraph we have that $[G : A \cap B] = [G : A][A : A \cap B] \leq [G : A][G : B]$ with equality if and only if $G = AB$. Since, the equality holds, $G = AB$.

   (b) If H is a proper subgroup of a finite group G, then the set union $\cup_{x \in G} x^{-1}Hx$ is not the whole of G.

   Solution: The number of conjugates of H equals the index of its normalizer that is $[G : N_H] \leq [G : H]$ since $H \subset N_H$. Let $|G| = n$ and $[G : H] = m$. Since all conjugates have at least the identity in common, $| \cup_{x \in G} x^{-1}Hx| \leq 1 + m(n/m - 1) = n - (m - 1) < n$, since H is proper and hence $m = [G : H] > 1$.

2. (a) List the Sylow subgroups of non-abelian groups of orders 21 and 39.

   Solution: A non-abelian group of order 21 has seven Sylow 3-groups and one Sylow 7-group. A non-abelian group of order 39 must have thirteen Sylow 3-groups and one Sylow 13-group.

   (b) Prove that there is no simple group of order 56.

   Solution: Let G be a group of order 56. Consider the Sylow 7-groups. If G is simple then it must have eight Sylow 7-groups. Any two of these can intersect only at the indentity. Hence their union must have cardinality $1 + 8(7 - 1) = 49$. Hence, there must be a unique Sylow 2-group which would imply that G is not simple.

3. Let A be a commutative ring with identity which is not a field. Prove that the following conditions are equivalent.

   (a) The sum of two non-invertible elements is non-invertible.

   (b) The non-invertible elements form a proper ideal.

(c) The ring A possesses a unique maximal ideal.

Give an example of a ring satisfying the above conditions and describe its unique maximal ideal.

Solution:
(a) $\Rightarrow$ (b): Let $a \in A$ be non-invertible. Then $-a$ is also non-invertible and 0 is non-invertible . Hence, the non-invertible elements form an additive subgroup of I of A. Morever, if $a \in I$ and $b \in A$, then $ba \in I$. Otherwise, let $c \in A$ such that $bac = 1$ which would imply that a is invertible. Since, $1 \notin I$ this shows that I is a proper ideal.

(b) $\Rightarrow$ (c): Let I be the ideal of the non-invertible elements and let M be a maximal ideal in A. Then clearly every element of M must be non-invertible, otherwise $1 \in M$. Hence, $M \subset I$. But since, I is proper, $M = I$.

(c) $\Rightarrow$ (a): Let M be the unique maximal ideal of A. Let x, y be two non-invertible elements of A. Then, $(x) \subset M, (y) \subset M$ since, $(x), (y)$ are proper ideals, and M is the unique maximal ideal. Then, $(x + y) \subset M$. Now, since $1 \notin M$, this implies that $x + y$ is also non-invertible.

Example: The ring $k[[x]]$ with the unique maximal ideal (x).

4. (a) Let α be the real positive fourth root of 2 and $i = \sqrt{-1}$. Find all intermediate fields in the extension $\mathbb{Q}(\alpha, i)$ over $\mathbb{Q}$.

   Solution: Handwritten in a separate page.

   (b) Let K be a finite field with p^n elements. Show that every element of K has a unique p-th root in K.

   Solution: Consider the Frobenius automorphism of $\sigma : K \to K$, sending $x \mapsto x^p$. Since, this is an automorphism of K, it is clear that for each $x \in K$, there exists a unique y such that $\sigma(y) = x$.

5. Recall that a square matrix is nilpotent if $A^p = 0$ for some $p > 0$.

   (a) If A is an $n \times n$ complex nilpotent matrix, then $A^n = 0$.

   Solution: Let $p > 0$ be the largest integer such that $A^p \neq 0$. Then, $A^p x \neq 0$ for some $x \in \mathbb{C}^n$. Then, the vectors $x, Ax, \ldots, A^p x$ are linearly independent. Otherwise, let $\sum_{0 \leq i \leq p} c_i A^i x = 0$ and not all $c_i = 0$. Let $1 \leq k < p$ be the least index such that $c_k \neq 0$. Then, $A^k x = \sum_{k < i \leq p} \frac{c_i}{c_k} A^i x$. Multiplying, by A^{p-k} we get, $A^p x = \sum_{k < i \leq p} \frac{c_i}{c_k} A^{p-k+i} x = 0$, because $A^{p+1} = 0$, a contradiction. Hence, $p < n$.

   (b) Prove that the characteristic polynomial of a nilpotent matrix A of order n is equal to λ^n.

Solution: The polynomial λ^n annihilates A and hence the minimal polynomial of A is $\lambda^m, 0 \leq m \leq n$. Thus, the characteristic polynomial is λ^n.

(c) Let A be a matrix of order n. Prove that A is nilpotent if and only if $tr(A^p) = 0$ for $p = 1, \ldots, n$.

One direction follows from the previous problem. In the other direction, first reduce the matrix A to its Jordan normal form and let $\lambda_1, \ldots, \lambda_k$ be its distinct non-zero eigenvalues. Let n_i be the sum of the orders of the Jordan blocks corresponding to the eigenvalue λ_i. Then, $tr(A^p) = \sum_{1 \leq i \leq k} n_i \lambda_i^p = 0, 1 \leq p \leq k$. Looking at these equations as a system equations in the unknowns n_i, we see that the determinant $\det((\lambda_i^p)_{i,p}) \neq 0$ (Vandermonde). Thus, the only solution is $n_1 = \cdots = n_k = 0$. Thus, all eigenvalues are zero and the characteristic polynomial of A is λ^n. Thus, $A^n = 0$ and A is nilpotent.