Department of Mathematics, University of California, Berkeley

YOUR 1 OR 2 DIGIT EXAM NUMBER ____

GRADUATE PRELIMINARY EXAMINATION, Part A Spring Semester 2017

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1. Please write your 1- or 2-digit exam number on this cover sheet and on **all** problem sheets (even problems that you do not wish to be graded).
2. Indicate below which six problems you wish to have graded. **Cross out** solutions you may have begun for the problems that you have not selected.
3. Extra sheets should be stapled to the appropriate problem at the upper right corner. Do not put work for problem p on either side of the page for problem q if $p \neq q$.
4. No notes, books, calculators or electronic devices may be used during the exam.

PROBLEM SELECTION

Part A: List the six problems you have chosen:

______ , ______ , ______ , ______ , ______ , ______

GRADE COMPUTATION (for use by grader—do not write below)

1A. ______	1B. ______	Calculus
2A. ______	2B. ______	Real analysis
3A. ______	3B. ______	Real analysis
4A. ______	4B. ______	Complex analysis
5A. ______	5B. ______	Complex analysis
6A. ______	6B. ______	Linear algebra
7A. ______	7B. ______	Linear algebra
8A. ______	8B. ______	Abstract algebra
9A. ______	9B. ______	Abstract algebra

Part A Subtotal: ______ Part B Subtotal: ______ Grand Total: ______

YOUR EXAM NUMBER ___

Please cross out this problem if you do not wish it graded

Problem 1A. *Score:*

Show that the following improper Riemann integrals exist and are equal:

$$\int_{-\infty}^{\infty} \frac{\sin(x)}{x} dx = \int_{-\infty}^{\infty} \frac{\sin^2(x)}{x^2} dx$$

Solution: Method 1:

$$\int_{-2L}^{2R} \frac{\sin(x)}{x} dx = \int_{-L}^{R} \frac{\sin(2x)}{x} dx = \int_{-L}^{R} \frac{2\sin(x)\cos(x)}{x} dx = \int_{-L}^{R} \frac{d}{dx}(\sin^2(x)) \frac{dx}{x} \ .$$

Integrate by parts:

$$= \frac{\sin^2(x)}{x} \Big|_{-L}^{R} + \int_{-L}^{R} \frac{\sin^2(x)}{x^2} dx.$$

As $L \to \infty$, $\sin^2(x)/x \to 0$ and the tail

$$\int_{-\infty}^{-L} \frac{\sin^2(x)}{x^2} dx \to 0.$$

Hence the improper integrals exist and are equal.

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Problem 2A. *Score:*

Suppose f is a function from the reals to the reals satisfying $2f(x) = f(2x)$ for all x.

(a) Prove that if f is differentiable at 0 then f is linear.

(b) Give an example of such a function f that is continuous but not linear.

Solution: (a) We have $f(0) = 0$. Suppose $f(x) = y$ for some $x \neq 0$. Then $f(x/2^n) = y/2^n$ so there are points arbitrarily close to 0 such that the graph of f intersects the line through 0 of slope y/x. So if f is differentiable at 0 with derivative c then $f(x)/x = y/x = c$. Since this holds for any x, f must be the function $y = cx$.

(b) The function $f(x) = xg(\log_2(x))$ for $x \neq 0$, $f(0) = 0$, where g is a continuous bounded function of period 1 such as $g(x) = \sin(2\pi x)$.

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Problem 3A. *Score:*

Suppose we have a continuous positive function $f : (0, \pi) \to (0, \infty)$ such that for all $x, y \in (0, \pi)$ we have

$$\int_x^y \frac{f(x)f(y)}{f^2(t)} dt = \sin(y - x).$$

(a) Show that $\sin(z - x)f(y) = \sin(y - x)f(z) + \sin(z - y)f(x)$.
(b) Find all possibilities for f.

Solution: For any $x, y, z \in (0, \pi)$ we have

$$\frac{\sin(z-x)}{f(x)f(z)} = \int_x^z \frac{dt}{f^2(t)} = \int_x^y \frac{dt}{f^2(t)} + \int_y^z \frac{dt}{f^2(t)} = \frac{\sin(y-x)}{f(x)f(y)} + \frac{\sin(z-y)}{f(y)f(z)}.$$

Multiplying both sides by $f(x)f(y)f(z)$ yields

$$\sin(z - x)f(y) = \sin(y - x)f(z) + \sin(z - y)f(x).$$

Note that $|z - x| < \pi$ and therefore $\sin(z - x) \neq 0$ unless $z = x$. So, fixing $x \neq z$ and letting y vary yields that f must be of the following form:

$$f(y) = a \sin(y + \theta).$$

for some $a > 0$ and $\theta \in [0, 2\pi)$. Since f was required to be positive on $(0, \pi)$, we find that $\theta = 0$. So

$$f(y) = a \sin(y).$$

We now claim that any function of this form solves the integral equation from the problem. To see this we compute

$$\int_x^y \frac{dt}{\sin^2(t)} = -\frac{\cos t}{\sin t}\Big|_x^y = \frac{\cos x}{\sin x} - \frac{\cos y}{\sin y} = \frac{\cos x \sin y - \cos y \sin x}{\sin x \sin y} = \frac{\sin(y-x)}{\sin x \sin y}.$$

So if $f(y) = a \sin(y)$, then

$$\int_x^y \frac{f(x)f(y)}{f^2(t)} dt = \sin x \sin y \int_x^y \frac{dt}{\sin^2(t)} = \sin(y - x).$$

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Problem 4A. *Score:*

The Weierstrass zeta function ζ is a meromorphic function satisfying

- $\zeta(z + \omega_1) = \zeta(z) + \eta_1$
- $\zeta(z + \omega_2) = \zeta(z) + \eta_2$
- The singularities of ζ are poles of residue 1 at the points $m\omega_1 + n\omega_2$ for $m, n \in \mathbb{Z}$

Here ω_1, ω_2, η_1, η_2 are complex constants with ω_2/ω_1 not real. Use Cauchy's residue theorem to prove Legendre's relation $\omega_2\eta_1 - \omega_1\eta_2 = \pm 2\pi i$ and express the sign in terms of ω_1 and ω_2.

Solution: Integrate ζ around a parallelogram containing 0 with sides parallel to the lines from 0 to ω_1 and ω_2. By the residue theorem the integral is $2\pi i$ times the sum of the residues, which is $2\pi i$. Suppose $\Im(\omega_2/\omega_1) > 0$. The sum of the integrals along two of the sides is $-\omega_1\eta_2$ (using the fact that $\zeta(z + \omega_2) = \zeta(z) + \eta_2$) and similarly the sum of the integrals over the other two sides is $\omega_2\eta_1$. Switching ω_1 and ω_2 changes a lot of signs, so we find $\omega_2\eta_1 - \omega_1\eta_2 = \pm 2\pi i$ where the sign is the sign of $\Im(\omega_2/\omega_1)$.

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Problem 5A. *Score:*

Suppose the coefficients of the power series

$$\sum_{n=0}^{\infty} a_n z^n$$

are given by the recurrence relation

$$a_0 = 1,\ a_1 = -1,\ 3a_n + 4a_{n-1} - a_{n-2} = 0,\ n = 2, 3, \ldots\ .$$

Find the radius of convergence of the series and the function to which it converges in its disc of convergence.

Solution: We can solve for $f(z)$ using the recurrence:

$$\begin{aligned}
3f(z) + 4zf(z) - z^2 f(z) &= 3\sum_{n=0}^{\infty} a_n z^n + 4\sum_{n=0}^{\infty} a_n z^{n+1} - \sum_{n=0}^{\infty} a_n z^{n+2} \\
&= 3a_0 + 3a_1 z + 3\sum_{n=2}^{\infty} a_n z^n + 4a_0 z + 4\sum_{n=2}^{\infty} a_{n-1} z^n - \sum_{n=2}^{\infty} a_{n-2} z^n \\
&= 3a_0 + 3a_1 z + 4a_0 z + \sum_{n=2}^{\infty} (3a_n + 4a_{n-1} - a_{n-2}) z^n \\
&= 3 + z\ .
\end{aligned}$$

Therefore

$$f(z) = \frac{3+z}{3+4z-z^2}\ .$$

This has poles where $z^2 - 4z - 3 = 0$; i.e., $z = 2 \pm \sqrt{7}$. The smallest absolute value of such a pole is $\sqrt{7} - 2$, so that is the radius of convergence of the series. (The series has positive radius of convergence because the coefficients, being solutions of a recurrence, grow at most exponentially. Therefore it is the Taylor series for the function at $z = 0$, and converges in the largest disc centered at $z = 0$ over which the function is holomorphic.)

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Problem 6A. *Score:*

Let A be an $n \times n$ matrix over the complex numbers. Let $e^A = 1 + A + A^2/2 + \cdots + A^m/m! + \cdots$. Show this series converges and $\det(e^A) = e^{Tr(A)}$.

Solution: Let M denote the max of the absolute values of the entries of A. Then one shows by induction that the max of the absolute values of the entries of A^m is at most $n^{m-1}M^m$. Hence, the entries of e^A are series of the form $\sum_{n\geq 0} a_n/n!$ such that $|a_n| \leq (nM)^m$ and so converge by the comparison test with e^{nM}.

Now suppose B is an invertible matrix such that $BAB^{-1} = C$ is upper triangular. Then $Be^AB^{-1} = 1 + BAB^{-1} + BA^2B^{-1}/2 + \cdots + BA^mB^{-1}/m! + \cdots = e^C$. But e^C is upper triangular and the diagonal entries are e^{c_i} where $c_1, \ldots, c_n$ are the diagonal entries of C. Hence $\det(e^A) = \det(e^C) = e^{\sum c_i} = e^{Tr(C)} = e^{Tr(A)}$.

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Problem 7A. *Score:*

Given two vectors x and y in $\mathbb{R}^n$ with $\|x\|_2 = \|y\|_2$, construct an orthogonal matrix Q such that $Qx = y$. Can there be such a matrix if $\|x\|_2 \neq \|y\|_2$?

Solution: Reflect across a plane P perpendicular to $x - y$. Let $u = (x - y)/\|x - y\|_2$. Then uu^T implements projection onto P, so

$$Q = I - 2uu^T$$

takes x into y:

$$Qx = x - 2uu^Tx = (\|x\|_2^2 - x^Ty)x - (\|x\|^2 - y^Tx)(x - y) = y.$$

Moreover,

$$Q^TQ = Q^2 = (I - 2uu^T)^2 = I - 4uu^Tuu^T + 4uu^T = I$$

so Q is orthogonal.

If $\|x\|_2 \neq \|y\|_2$ there can be no Q, since orthogonal matrices define isometries.

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Problem 8A. *Score:*

Show that for each integer $p \geq 0$ the sum

$$S_p(n) = \sum_{k=0}^{n} k^p$$

is a polynomial of degree $p+1$ in the variable n.

Solution: Define the backward difference operator Δ so that

$$\Delta S_p(n) = S_p(n) - S_{p-1}(n) = n^p - (n-1)^p = \sum_{k=0}^{p-1} \binom{n}{k} n^k (-1)^{n-k+1}$$

is a polynomial of degree $p-1$. Similarly $\Delta^k S_p(n)$ is a polynomial of degree $p+1-k$ and $\Delta^{p+2} S_p(n) = 0$. This linear homogeneous difference equation of order $p+2$ has $p+2$ linearly independent solutions. Among them are all polynomials in n of degree $p+1$. Since these polynomials form a subspace of dimension $p+2$, they form the full solution space of the difference equation. Hence the solution $S_p(n)$ determined by $p+2$ known initial values is a polynomial in n of degree $p+1$.

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Problem 9A. *Score:*

The Bell number P_n is the number of partitions of a set of n elements into disjoint nonempty subsets, so for example $\{1,2,3\} = \{1\} \cup \{2\} \cup \{3\} = \{1,2\} \cup \{3\} = \{2,3\} \cup \{1\} = \{1,3\} \cup \{2\}$ and $P_3 = 5$. Show that

$$\frac{P_n}{n!} \to 0$$

as $n \to \infty$.

Solution: For each partition let k be the cardinality of the subset S containing 1. There are $\binom{n-1}{k-1}$ ways to choose the other $k-1$ elements in S, and P_{n-k} ways to partition the other $n-k$, so

$$P_n = \sum_{k=1}^{n} \binom{n-1}{k-1} P_{n-k}$$

where $P_0 = 1$. E.g. $P_1 = 1$, $P_2 = 2$, $P_3 = 5$, and so forth. Shifting gives

$$Q_{n+1} = \frac{P_{n+1}}{(n+1)!} = \frac{1}{n+1} \sum_{k=0}^{n} \frac{1}{k!} Q_{n-k}.$$

Hence

$$Q_{n+1} \le \frac{1}{n+1} \sum_{k=0}^{n} \frac{1}{k!} \max_{0 \le k \le n} Q_k \le \frac{e}{n+1} \max_{0 \le k \le n} Q_k.$$

For $n \ge 2$, $e < n+1$ so (a) the maximum of the Q's cannot increase and (b) $Q_n \to 0$ as $n \to \infty$.

Department of Mathematics, University of California, Berkeley

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GRADUATE PRELIMINARY EXAMINATION, Part B Spring Semester 2017

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PROBLEM SELECTION

Part B: List the six problems you have chosen:

______ , ______ , ______ , ______ , ______ , ______

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Problem 1B. *Score:*

Find all differentiable functions $f : \mathbb{R} \to \mathbb{R}$ with the property that

$$f'(x) = \frac{f(x+h) - f(x-h)}{2h}$$

for all $x \in \mathbb{R}$ and all $h \neq 0$. (Hint: multiply both sides by $2h$.)

Solution: Note first that for fixed $h \neq 0$ the right-hand side of the equation is differentiable in x. So $f(x)$ is twice differentiable. Iterating this argument yields that $f(x)$ is even smooth.

Now if we multiply both sides of the equation by $2h$, then we get

$$2h \cdot f'(x) = f(x+h) - f(x-h).$$

If we differentiate this equation by h (for some fixed x), we obtain

$$2f'(x) = f'(x+h) + f'(x-h).$$

Differentiating this equation once more by h gives

$$0 = f''(x+h) - f''(x-h).$$

Since this equation holds for all $x \in \mathbb{R}$ and $h \neq 0$, we can conclude that $f''(x)$ is constant. Therefore, $f(x)$ must be of the form

$$f(x) = ax^2 + bx + c$$

for some fixed $a, b, c \in \mathbb{R}$.

We claim that, conversely, any function of this form satisfies the desired equation. To see this, observe that

$$\begin{aligned}\frac{f(x+h) - f(x-h)}{2h} &= \frac{a(x+h)^2 + b(x+h) + c - a(x-h)^2 - b(x-h) - c}{2h} \\ &= \frac{4axh + 2bh}{2h} = 2ax + b = f'(x).\end{aligned}$$

YOUR EXAM NUMBER ___
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Problem 2B. *Score:*

Suppose $f : [-1, 1] \to \mathbb{C}$ is a continuous complex-valued function, and for all non-negative integers n

$$\int_{-1}^{1} x^n f(x)dx = 0.$$

Prove that $f = 0$.

Solution: Method 1: By taking linear combinations,

$$\int_{-1}^{1} f(x)P(x)dx = 0$$

whenever P is a polynomial. Given $\epsilon > 0$, the Weierstrass Approximation Theorem provides a polynomial P with

$$|f(x) - P(x)| \leq \epsilon$$

for $-1 \leq x \leq 1$. Then

$$\int_{-1}^{1} f(x)^2 dx = \int_{-1}^{1} f(x)(f(x) - P(x) + P(x))dx \leq \epsilon \int_{-1}^{1} |f(x)|dx.$$

Since $\epsilon > 0$ was arbitrary,

$$\int_{-1}^{1} f(x)^2 dx = 0.$$

Since f is continuous, $f = 0$.

Method 2: By Taylor expansion, all the Fourier coefficients

$$\hat{f}(k) = \int_{-1}^{1} e^{i\pi kx} f(x)dx = 0.$$

By the uniqueness of Fourier series coefficients, we must have $f = 0$ almost everywhere, and since f is continuous we must have $f = 0$.

Problem 3B. *Score:*

The error of a quadrature rule with $p+1$ distinct points x_j, weights w_j is

$$E_p(f) = \int_a^b f(x)dx - \sum_{j=0}^{p} w_j f(x_j).$$

Suppose that $E_p(f) = 0$ whenever f is a polynomial of degree $\leq q$. Show that $q \leq 2p+1$ and if $q \geq 2p$ then $w_j > 0$ for all j.

Solution: Define a polynomial f of degree $2p+2$ by

$$f(x) = \prod_{j=0}^{p} (x - x_j)^2,$$

so that

$$E_p(f) = \int_a^b f(x)dx > 0.$$

Since $\deg f = 2p+2$, we must have $q \leq 2p+1$.

Let

$$f_j(x) = \prod_{k \neq j} (x - x_j)^2.$$

Since $\deg f = 2p$, we must have $E_p(f) = 0$. Hence

$$\int_a^b f(x)dx = w_j \prod_{k \neq j} (x_k - x_j)^2$$

and

$$w_j = \frac{\int_a^b f(x)dx}{\prod_{k \neq j}(x_k - x_j)^2} > 0.$$

YOUR EXAM NUMBER ___

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Problem 4B. *Score:*

Given n distinct points $z_j \in \mathbb{C}$ and n values $f_j \in \mathbb{C}$, show that there is a unique polynomial P of degree at most $n-1$ such that

$$P(z_j) = f_j$$

for $1 \le j \le n$.

Solution: Define n polynomials P_j of degree $n-1$ by

$$P_j(z) = \prod_{k \neq j} \frac{z - z_k}{z_j - z_k},$$

so that $P_j(z_k) = \delta_{jk}$. Then

$$P(z) = \sum_{j=1}^{n} f_j P_j(z)$$

satisfies $P(z_j) = f_j$ for $1 \le j \le n$.

Since the solution $a \in \mathbb{C}^n$ of the $n \times n$ linear system

$$\sum_{j=0}^{n-1} a_j z_k^j = f_k$$

exists for every right-hand side vector $f \in \mathbb{C}^n$, the fundamental theorem of linear algebra guarantees uniqueness.

YOUR EXAM NUMBER ____

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Problem 5B. *Score:*

Write all values of i^i in the form $a + bi$.

Solution: We have $\log i = \log |i| + i \arg i = 0 + i\pi/2$ (using the main branch of the log function). Taking all branches, we have $\log i = i(\pi/2 + 2n\pi)$, $n \in \mathbb{Z}$. Therefore

$$i^i = e^{i \log i} = e^{i^2(\pi/2+2n\pi)} = e^{-(\pi/2+2n\pi)} + 0i \ , \quad n \in \mathbb{Z} \ .$$

YOUR EXAM NUMBER ____

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Problem 6B. *Score:*

Let D be the unit disk in the complex plane $\mathbb{C}$, $f : D \to \mathbb{C}$ an analytic function with

$$|f^{(k)}(0)| \leq M$$

for all $k \geq 0$, and let $t_p \in D$, $s_p \in D$ for $1 \leq p \leq n$. For each $n \geq 1$ define $A_{ij} = f(t_i s_j)$ for $1 \leq i, j \leq n$. For each $r \geq 1$ find an $n \times n$ matrix B with rank $\leq r$ and

$$|A_{ij} - B_{ij}| \leq \frac{2M}{r!}$$

for $1 \leq i, j \leq n$.

Solution: By Taylor expansion,

$$B_{ij} = \sum_{k=0}^{r-1} \frac{(t_i s_j)^k}{k!} f^{(k)}(0)$$

satisfies

$$|A_{ij} - B_{ij}| \leq \sum_{k=r}^{\infty} \frac{M}{k!} \leq \frac{2M}{r!}$$

by the geometric series.

YOUR EXAM NUMBER ____

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Problem 7B. *Score:*

Suppose R is an invertible upper triangular complex matrix and A is symmetric. Find an explicit formula for the entries of the upper triangular matrix E satisfying

$$E^T R + R^T E = A$$

and show that your solution is unique. Hint: Multiply by R^{-1T} and R^{-1}.

Solution: Multiply by R^{-1T} and R^{-1} to get

$$R^{-1T} E^T + E R^{-1} = R^{-T} A R^{-1}$$

Since invertible upper triangular matrices form a group, ER^{-1} is upper triangular and $R^{-T}E^T$ is lower triangular. Hence equating the two sides entry by entry shows that

$$E = \mathrm{uph}(R^{-1T} A R^{-1})$$

where $\mathrm{uph}(B)$ is the upper triangle of B with the diagonal entries halved. Since E has $n(n+1)/2$ entries satisfying $n(n+1)/2$ linear equations to which we have shown a solution exists for every right-hand side A, the fundamental theorem of linear algebra guarantees its uniqueness.

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Problem 8B. *Score:*

Find a product of cyclic groups of prime power order isomorphic to $(\mathbb{Z}/1000000\mathbb{Z})^*$ (the group of units of the ring of integers mod 1000000).

Solution: $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2^4\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/5^5\mathbb{Z}$.

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Problem 9B. *Score:*

Let S_9 denote the group of permutations of 9 objects.

(a) Exhibit an element of S_9 of order 20.

(b) Prove that no element of S_9 has order 18.

Solution: (a) $(1\,2\,3\,4)(5\,6\,7\,8\,9)$ has order 20, because it is a product of a 4-cycle and a (disjoint) 5-cycle. They commute because they are disjoint, and they have orders 4 and 5, respectively, so their product has order 20.

(b) Suppose that $\sigma \in S_9$ has order 18. Let $n_1, \ldots, n_r$ be the orders of its nontrivial cycles in a disjoint cycle decomposition. Then $n_i > 1$ for all i, their lcm is 18, and their sum is at most 9.

This is impossible. Indeed, At least one n_i is a multiple of 9, so it must equal 9, and there can then be no other n_i in the sequence because the sum is ≤ 9. But $\{9\}$ does not have lcm 18, a contradiction.