Preliminary Exam - Spring 1977

Problem 1 *Suppose f is a differentiable function from the reals into the reals. Suppose $f'(x) > f(x)$ for all $x \in \mathbb{R}$, and $f(x_0) = 0$. Prove that $f(x) > 0$ for all $x > x_0$.*

Problem 2 *Suppose that f is a real valued function of one real variable such that*

$$\lim_{x \to c} f(x)$$

exists for all $c \in [a, b]$. Show that f is Riemann integrable on $[a, b]$.

Problem 3

1. *Evaluate $P_{n-1}(1)$, where $P_{n-1}(x)$ is the polynomial*

$$P_{n-1}(x) = \frac{x^n - 1}{x - 1}.$$

2. *Consider a circle of radius 1, and let $Q_1, Q_2, \ldots, Q_n$ be the vertices of a regular n-gon inscribed in the circle. Join Q_1 to $Q_2, Q_3, \ldots, Q_n$ by segments of a straight line. You obtain $(n-1)$ segments of lengths $\lambda_2, \lambda_3, \ldots, \lambda_n$. Show that*

$$\prod_{i=2}^{n} \lambda_i = n.$$

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Problem 4 *Prove the Fundamental Theorem of Algebra: Every nonconstant polynomial with complex coefficients has a complex root.*

Problem 5 *Let $\mathbf{F} \subset \mathbf{K}$ be fields, and a and b elements of $\mathbf{K}$ which are algebraic over $\mathbf{F}$. Show that $a + b$ is algebraic over $\mathbf{F}$.*

Problem 6 *Let G be the collection of 2×2 real matrices with nonzero determinant. Define the product of two elements in G as the usual matrix product.*

1. *Show that G is a group.*

2. *Find the* center Z *of* G*; that is, the set of all elements* z *of* G *such that* $az = za$ *for all* $a \in G$.

3. *Show that the set* O *of real orthogonal matrices is a subgroup of* G *(a matrix is* orthogonal *if* $AA^t = I$*, where* A^t *denotes the transpose of* A*). Show by example that* O *is not a normal subgroup.*

4. *Find a nontrivial homomorphism from* G *onto an abelian group.*

Problem 7 *A matrix of the form*

$$\begin{pmatrix} 1 & a_0 & a_0^2 & \dots & a_0^n \\ 1 & a_1 & a_1^2 & \dots & a_1^n \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & a_n & a_n^2 & \dots & a_n^n \end{pmatrix}$$

where the a_i *are complex numbers, is called a Vandermonde matrix.*

1. *Prove that the Vandermonde matrix is invertible if* $a_0, a_1, \dots, a_n$ *are all different.*

2. *If* $a_0, a_1, \dots, a_n$ *are all different, and* $b_0, b_1, \dots, b_n$ *are complex numbers, prove that there is a unique polynomial* f *of degree* n *with complex coefficients such that* $f(a_0) = b_0,\ f(a_1) = b_1, \dots,\ f(a_n) = b_n$.

Problem 8 *Find a list of real matrices, as long as possible, such that*

- *the characteristic polynomial of each matrix is* $(x-1)^5(x+1)$,
- *the minimal polynomial of each matrix is* $(x-1)^2(x+1)$,
- *no two matrices in the list are similar to each other.*

Problem 9 *Find the solution of the differential equation*

$$y'' - 2y' + y = 0,$$

subject to the conditions

$$y(0) = 1, \quad y'(0) = 1.$$

Problem 10 *In* $\mathbb{R}^2$*, consider the region* $\mathcal{A}$ *defined by* $x^2 + y^2 > 1$*. Find differentiable real valued functions* f *and* g *on* $\mathcal{A}$ *such that* $\partial f/\partial x = \partial g/\partial y$ *but there is no real valued function* h *on* $\mathcal{A}$ *such that* $f = \partial h/\partial y$ *and* $g = \partial h/\partial x$*.*

Problem 11 *Let the sequence* $a_0, a_1, \ldots$ *be defined by the equation*

$$1 - x^2 + x^4 - x^6 + \cdots = \sum_{n=0}^{\infty} a_n (x-3)^n \quad (0 < x < 1).$$

Find

$$\limsup_{n\to\infty} \left(|a_n|^{\frac{1}{n}} \right).$$

Problem 12 *Let* p *be an odd prime. Let* $Q(p)$ *be the set of integers* a*,* $0 \leqslant a \leqslant p-1$*, for which the congruence*

$$x^2 \equiv a \pmod p$$

has a solution. Show that $Q(p)$ *has cardinality* $(p+1)/2$*.*

Problem 13 *Consider the family of square matrices* $A(\theta)$ *defined by the solution of the matrix differential equation*

$$\frac{dA(\theta)}{d\theta} = BA(\theta)$$

with the initial condition $A(0) = I$*, where* B *is a constant square matrix.*

1. *Find a property of* B *which is necessary and sufficient for* $A(\theta)$ *to be* orthogonal *for all* θ*; that is,* $A(\theta)^t = A(\theta)^{-1}$*, where* $A(\theta)^t$ *denotes the transpose of* $A(\theta)$*.*

2. *Find the matrices* $A(\theta)$ *corresponding to*

$$B = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

   and give a geometric interpretation.

Problem 14 *A square matrix* A *is nilpotent if* $A^k = 0$ *for some positive integer* k*.*

1. *If A and B are nilpotent, is $A + B$ nilpotent?*

2. *Prove: If A and B are nilpotent matrices and $AB = BA$, then $A + B$ is nilpotent.*

3. *Prove: If A is nilpotent then $I + A$ and $I - A$ are invertible.*

Problem 15 *Let $f(z)$ be a nonconstant meromorphic function. A complex number w is called a period of f if $f(z + w) = f(z)$ for all z.*

1. *Show that if w_1 and w_2 are periods, so are $n_1 w_1 + n_2 w_2$ for all integers n_1 and n_2.*

2. *Show that there are, at most, a finite number of periods of f in any bounded region of the complex plane.*

Problem 16 *Evaluate*

$$\int_{-\infty}^{\infty} \frac{\cos nx}{x^4 + 1} \, dx \ .$$

Problem 17 *Let $\mathbb{Q}_+$ be the multiplicative group of positive rational numbers.*

1. *Is $\mathbb{Q}_+$ torsion free?*

2. *Is $\mathbb{Q}_+$ free?*

Problem 18

1. *In $\mathbb{R}[x]$, consider the set of polynomials $f(x)$ for which $f(2) = f'(2) = f''(2) = 0$. Prove that this set forms an ideal and find its monic generator.*

2. *Do the polynomials such that $f(2) = 0$ and $f'(3) = 0$ form an ideal?*

Problem 19 *Suppose that $u(x, t)$ is a continuous function of the real variables x and t with continuous second partial derivatives. Suppose that u and its first partial derivatives are periodic in x with period 1, and that*

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2}.$$

Prove that

$$E(t) = \frac{1}{2} \int_0^1 \left(\left(\frac{\partial u}{\partial t} \right)^2 + \left(\frac{\partial u}{\partial x} \right)^2 \right) dx$$

is a constant independent of t.

Problem 20 *Let* $h : [0,1) \to \mathbb{R}$ *be a function defined on the half-open interval* $[0,1)$*. Prove that if* h *is uniformly continuous, there exists a unique continuous function* $g : [0,1] \to \mathbb{R}$ *such that* $g(x) = h(x)$ *for all* $x \in [0,1)$*.*