Department of Mathematics, University of California, Berkeley

YOUR 1 OR 2 DIGIT EXAM NUMBER ____

GRADUATE PRELIMINARY EXAMINATION, Part A — Spring Semester 2019

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1. Please write your 1- or 2-digit exam number on this cover sheet and on **all** problem sheets (even problems that you do not wish to be graded).
2. Indicate below which six problems you wish to have graded. **Cross out** solutions you may have begun for the problems that you have not selected.
3. Extra sheets should be stapled to the appropriate problem at the upper right corner. Do not put work for problem p on either side of the page for problem q if $p \neq q$.
4. No notes, books, calculators or electronic devices may be used during the exam.

PROBLEM SELECTION

Part A: List the six problems you have chosen:

______ , ______ , ______ , ______ , ______ , ______

GRADE COMPUTATION (for use by grader—do not write below)

1A. ______	1B. ______	Calculus
2A. ______	2B. ______	Real analysis
3A. ______	3B. ______	Real analysis
4A. ______	4B. ______	Complex analysis
5A. ______	5B. ______	Complex analysis
6A. ______	6B. ______	Linear algebra
7A. ______	7B. ______	Linear algebra
8A. ______	8B. ______	Abstract algebra
9A. ______	9B. ______	Abstract algebra

Part A Subtotal: ______ Part B Subtotal: ______ Grand Total: ______

YOUR EXAM NUMBER ____

Please cross out this problem if you do not wish it graded

Problem 1A. *Score:*

Let y be a solution of $y''' - y = 0$ such that $y(t) \to 0$ as $t \to \infty$. Show that $y(0) + y'(0) + y''(0) = 0$.

Solution:

Problem 2A. *Score:*

Let $f : \mathbb{R} \to \mathbb{R}$ be bounded and continuously differentiable. Show that every solution y of $y' = f(y)$ is monotone.

Solution:

YOUR EXAM NUMBER ___
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Problem 3A. *Score:*

Let f be a twice continuously differentiable function on $[0,1]$ such that $f(0) = f(1) = 0$. Prove that

$$\max_{x\in[0,1]} |f(x)| \leq \frac{1}{8} \max_{x\in[0,1]} |f''(x)|,$$

and find an example where equality holds.

Solution:

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Problem 4A. *Score:*

Evaluate

$$I = \int_{-\infty}^{\infty} \frac{x \sin x}{(1+x^2)^2}\ dx.$$

Solution:

YOUR EXAM NUMBER ____

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Problem 5A. *Score:*

Find the number of complex roots of $e^z = 3z^6$ with $|z| < 1$ that have positive imaginary part.

Solution:

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Problem 6A. *Score:*

Let n be a positive integer and let a be a complex number. Prove that $a^n = 1$ if and only if there are invertible n by n complex matrices X, Y such that $YX = aXY$.

Solution:

Problem 7A. *Score:*

Let A be a complex $n \times n$ matrix satisfying $A^{37} = I$. Show that A is diagonalizable.

Solution:

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Problem 8A. *Score:*

Show that $F : \mathbb{C}^3 \to \mathbb{C}^3$ defined by

$$F(u, v, w) = (-u - v - w, uv + uw + vw, -uvw)$$

is surjective but not injective.

Solution:

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Problem 9A. *Score:*

Let S be a countable set of real numbers. Show that there are function $g_n : \mathbb{N} \to \mathbb{N}$ such that if $f : \mathbb{N} \to \mathbb{N}$ is a function from $\mathbb{N}$ to $\mathbb{N}$ with $f(n+1) > g_n(f(n))$ for all n, then $A = \sum_{n=1}^{\infty} \frac{1}{f(n)}$ converges to a real that is not in the set S.

Solution:

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GRADUATE PRELIMINARY EXAMINATION, Part B Spring Semester 2019

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PROBLEM SELECTION

Part B: List the six problems you have chosen:

______ , ______ , ______ , ______ , ______ , ______

YOUR EXAM NUMBER ____
Please cross out this problem if you do not wish it graded

Problem 1B. *Score:*

Evaluate

$$I = \int_0^{\pi/2} \frac{\sin x \cos x}{\sin^4 x + \cos^4 x} dx$$

Solution:

YOUR EXAM NUMBER ____

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Problem 2B. *Score:*

For $t \geq 0$ let

$$F(t) = \int_0^t \exp(-x^2)dx$$

and

$$G(t) = \int_0^1 \frac{\exp(-t^2(1+x^2))}{1+x^2}dx.$$

Show that $F(t)^2 + G(t)$ is constant and deduce the value of $F(\infty)$.

Solution:

YOUR EXAM NUMBER ____
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Problem 3B. *Score:*

Show that $x_{n+1} = (1 + x_n)^{-1}$ converges and find its limit for any $x_0 > 0$.

Solution:

YOUR EXAM NUMBER ___
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Problem 4B. *Score:*

Let c_0, c_1, $\ldots$, c_{n-1} be complex numbers. Prove that all the zeroes of the polynomial

$$z^n + c_{n-1}z^{n-1} + \cdots + c_1 z + c_0$$

lie in the open disc with center 0 and radius

$$1 + |c_{n-1}| + \cdots + |c_1| + |c_0| \, .$$

Solution:

YOUR EXAM NUMBER ___

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Problem 5B. *Score:*

If $f(z)$ is analytic in the open disc $\mathbb{D} = \{z : |z| < 1\}$, and if $|f(z)| < 1/(1 - |z|)$ for all $z \in \mathbb{D}$, show that

$$\left| \frac{f^{(n)}(0)}{n!} \right| \leq (n+1) \left(1 + \frac{1}{n} \right)^n < e(n+1) \; .$$

Solution:

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Problem 6B. *Score:*

Let $\mathbb{Z}_2$ be the ring of integers mod 2. Prove the following identity in $\mathbb{Z}_2[x_1, \ldots, x_n]$:

$$\det \begin{bmatrix} x_1 & \ldots & x_n \\ x_1^2 & \ldots & x_n^2 \\ & \ldots & \\ x_1^{2^{n-1}} & \ldots & x_n^{2^{n-1}} \end{bmatrix} = \prod_{(a_1,\ldots,a_n)\neq(0,\ldots,0)} (a_1x_1 + \cdots + a_nx_n),$$

where $(a_1 \ldots a_n)$ run all non-zero values in $\mathbb{Z}_2^n$.

Solution:

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Problem 7B. *Score:*

Is it true that elements of the group $GL_2^+(\mathbb{R})$ of real 2×2-matrices with positive determinant are conjugate in $GL_2^+(\mathbb{R})$ if and only if the matrices are similar (conjugate in $GL_2(\mathbb{R})$)? Either prove this or give a counterexample.

Solution:

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Problem 8B. *Score:*

Let R be a ring (possibly non-commutative, possibly without an identity 1) in which every element is idempotent (this means that for all $a \in R$, $a^2 = a$). Show that R has characteristic 2 ($2a = 0$ for all a) and is commutative.

Solution:

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Problem 9B. *Score:*

Recall that S_6 and A_6 are the symmetric group and alternating group on 6 letters, respectively.

Prove or give a counterexample (with explanation): For every $\sigma \in A_6$ there is a $\tau \in S_6$ such that $\tau^2 = \sigma$.

Solution: