Topology Qualifying Exam Workshop

May 2020

Worksheet 2A

Theme: Homeomorphism invariants: Separation properties (T_0, T_1, Hausdorff, regular, normal), countability properties; connectedness, path connectedness, components; compactness, metrizability. Applications.

Part 1 - 5-10 minutes

Warm-up:

1. What are the properties $T_0 - T_4$? What other names are there for $T_2 - T_4$?
2. Pick an $n \in \{0, 1, 2, 3\}$, and give an example of a space that is T_n, but not T_{n+1}.
3. What property on a space guarantees that compact sets are closed? What property on a space guarantees that closed sets are compact?
4. Show that if there is a path between two points, they lie in the same connected component. Why does this show that a path connected space is connected?
5. Optional: Explain why the "topologist's sine wave" $X = \{(0,0)\} \cup \{(x, \sin(1/x)) : x > 0\}$ is an example of a space which is connected but not path connected.

Part 2 - 1 hour and 40 minutes

Try these problems:

1. (May 2015) If $X = \mathbb{R}^n$ with the usual Euclidean topology and $\mathcal{U} \subseteq X$ is an open, connected subspace of X, show that $\mathcal{U}$ is also path connected.
2. (January 2017) A space X is *completely regular* if X is T_1 and whenever C is closed in X and $p \in X \setminus C$, then there is a continuous function $f : X \to [0,1]$ such that $f(p) = 0$ and $f(C) = \{1\}$.
   (a) Let X be a T_1 space with topology $\mathcal{T}$ generated by a subbasis $\mathcal{S}$. Show that X is completely regular if and only if for every $p \in X$ and subbasis element $U \in \mathcal{S}$ containing p, there is a continuous function $f : X \to [0,1]$ such that $f(p) = 0$ and $f(X \setminus U) = \{1\}$.
   (b) Show that if X and Y are completely regular spaces, then so is the product space $X \times Y$. (Hint: use the result in (a).)
3. (May 2018) Show that a closed subset $A \subseteq X$ of a normal (i.e. T_1 and T_4) space $(X, \mathcal{T})$, with the subspace topology, is normal.
4. (May 2019) Show that a topological space $(X, \mathcal{T})$ is *Hausdorff* if and only if the diagonal $\Delta = \{(x, x) : x \in X\}$ is a closed subset of $X \times X$, where $X \times X$ is equipped with the product topology.
5. (May 2019) Recall that a topological space X is *second countable* if there exists a countable basis for the topology on X, and X is *separable* if it has a countable dense subset.
   (a) Show that if $(X, \mathcal{T})$ has a metric topology, from the metric $d : X \times X \to \mathbb{R}$, and X is separable, then X is second countable.
   (b) Show that $\mathbb{R}$, with the lower limit topology $\mathcal{T}_\ell$[1] is not metrizable, i.e., $\mathcal{T}_\ell$ is not the topology from any metric on $\mathbb{R}$.

[1]This is the topology generated by half-open intervals $[a, b)$.

Part 3 - 5-10 minutes

Wrap-up discussion with everyone:

1. Which of these problems did you find easy? Hard?
2. Which topics in the theme do you need to review more?