Topology Qualifying Exam Workshop

May 2020

Worksheet 2

Theme: Continuous deformations: Retraction, deformation retraction, contractible, mapping cylinder, homotopic maps, homotopy type.

Also more homeomorphism invariants because those are heavily emphasized on the quals.

Part 1 - 5-10 minutes

Warm-up:

1. Determine whether the following spaces X admit a retraction and/or a deformation retraction onto the designated subspace Y.
   (a) $X = [0, 1]$, $Y = \{0\}$.
   (b) $X = [0, 1]$, $Y = \{0, 1\}$.
   (c) $X = S^1$, $Y = \{y\}$, any singleton.
2. Construct an explicit deformation retraction from $\mathbb{R}^n \setminus \{0\}$ onto S^{n-1}.
3. Let $X = D^2$, $Y = \{y\}$ a singleton, and $f : X \to Y$ be the constant function. Construct and describe the mapping cylinder.

Part 2 - 1 hour and 40 minutes

Try these problems:

1. (May 2015) Let $f : X \to Y$ be a continuous map, and suppose that there are continuous maps $g, h : Y \to X$ with $f \circ g \simeq \mathrm{id}_Y$ and $h \circ f \simeq \mathrm{id}_X$.
   (a) Show that $g \simeq h$.
   (b) Show, moreover, that f is a homotopy equivalence.
2. (May 2018) Show that if X and Y are topological spaces and $X \times Y$, with the product topology, is contractible, then both X and Y are contactible.
3. (January 2019) Show that if $f, g : (X, \mathcal{T}) \to (Y, \mathcal{T}')$ and $h, k : (Y, \mathcal{T}') \to (Z, \mathcal{T}'')$ are continuous and $f \simeq g$ and $h \simeq k$, then $h \circ f \simeq k \circ g$. Show, however, that the converse ($h \circ f \simeq k \circ g$ implies $f \simeq g$ and $h \simeq k$) need not be true.
4. (June 2010) Let $Z = X \cup Y$, for X and Y connected subspaces of Z with $X \cap Y = \emptyset$. Let $x_0 \in X$ and $y_0 \in Y$. Let $\sim$ be the equivalence relation generated by the equivalence $x_0 \sim y_0$. Show that the quotient space $Z/\sim$ is connected.
5. (June 2011) Suppose that A, B are disjoint, compact subspaces of the Hausdorff topological space X. Prove that there are open subsets U, V of X such that $A \subset U$, $B \subset V$, and $U \cap V = \emptyset$.

Part 3 - 5-10 minutes

Wrap-up discussion with everyone:

1. Which of these problems did you find easy? Hard?
2. Which topics in the theme do you need to review more?