Topology Qualifying Exam Workshop

May 2020

Worksheet 4A

Theme: Homology: Simplicial homology, singular homology, induced homomorphism, homotopy invariance; exact sequence, long exact homology sequence, Mayer-Vietoris Theorem. Applications.

Part 1 - 5-10 minutes

Warm-up:

1. Give a Δ-complex structure for a standard 2-simplex Δ and write down its chain complex. Explicitly describe the boundary homomorphisms.
2. Give a Δ-complex structure for S^1., and use that to compute the homology groups for S^1.
3. What is the difference between the reduced homology and (non-reduced) homology?
4. Let X be a topological space and $A, B \subset X$ open subspaces with $X = A \cup B$. What does the Mayer-Vietoris theorem say about the homology groups of X and its subspaces?

Part 2 - 1 hour and 40 minutes

Try these problems:

1. (May 2015) Compute the (reduced) singular homology groups of the space $X = S_1 \times (S_1 \vee S_1)$, which can be thought of as two copies of $S_1 \times S_1$ glued together along their copies of $S_1 \times \{x_0\}$. [You may use your knowledge of the homology groups of $T^2 = S_1 \times S_1$ in your calculations.]
2. (January 2017) If X is a Δ-complex with at most one k-simplex for each $0 \leq k \leq 5$, show that each of the homology groups $H_k(X)$ must be cyclic for $0 \leq k \leq 5$. Can all of these groups be non-trivial? Explain why or why not.
3. (May 2018) Let Δ_2 and Δ_2' be distinct 2-simplices, and let X be the quotient space obtained by identifying the six vertices of $\Delta_2 \cup \Delta_2'$ to a single point. Identify a Δ-complex structure for X and compute the simplicial homology groups $H_n^{\Delta}(X)$ for all n.
4. (January 2019) Recalling that $S^2 = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1\}$, define the *equator* of S^2 to be $\{(x, y, z) \in S^2 : z = 0\}$, so that the equator is homeomorphic to S^1. Let Z_1 and Z_2 be disjoint copies of the 2-sphere S^2, let f be a homeomorphism from the equator of Z_1 to the equator of Z_2, and let Z be the quotient space obtained from $Z_1 \cup Z_2$ by identifying the equator of Z_1 to the equator of Z_2 via f. Find a Δ-complex structure on the space Z, and compute the simplicial homology groups of Z.
5. (May 2019) For a topological space X, the *cone on* X, cX, is the quotient space of $X \times [0, 1]$ under the equivalence relation $(x, s) \sim (y, t)$ iff either $(x, s) = (y, t)$ or $s = t = 1$. The *suspension of* X ΣX, is the union of two copies of cX along $X \times \{0\}$ realized as the quotient space of $X \times [0, 1]$ under the equivalence relation $(x, s) \sim (y, t)$ iff either $(x, s) = (y, t)$ or $s = t = 1$ or $s = t = 0$.
   (a) Show that the cone on X is contractible.
   (b) Use a Mayer-Vietoris sequence to show that for every $n \geq 1$ we have that $\tilde{H}_n(\Sigma X) \cong \tilde{H}_{n-1}(X)$.

Part 3 - 5-10 minutes

Wrap-up discussion with everyone:

1. Which of these problems did you find easy? Hard?
2. Which topics in the theme do you need to review more?