Topology Qualifying Exam Workshop

May 2020

Worksheet 5

Part 1 - 2 hours

Try these problems. On the exam you will be asked to complete three of the four questions in 3 hours.

1. (May 2013) Let X_α be non-empty topological spaces and suppose that $X = \prod_\alpha X_\alpha$ is endowed with the product topology.
   (a) Prove that each projection map π_α is continuous and open.
   (b) Prove that X is Hausdorff if and only if each space X_α is Hausdorff.
2. (May 2013)
   (a) Prove that every compact subspace of a Hausdorff space is closed. Show by example that the Hausdorff hypothesis cannot be removed.
   (b) Prove that every compact Hausdorff space is normal.
3. (May 2013) Define an equivalence relation $\sim$ on $\mathbb{R}^2$ by $(x_1, y_1) \sim (x_2, y_2)$ if and only if $x_1^2+y_1^2 = x_2^2+y_2^2$.
   (a) Identify the quotient space $X = \mathbb{R}^2/\sim$ as a familiar space and prove that it is homeomorphic to this familiar space.
   (b) Determine whether the natural map $p : \mathbb{R}^2 \to X$ is a covering map. Justify your answer.
4. (May 2013) A space is *locally connected* if for each point $x \in X$ and every neighborhood U of x, there is a connected neighborhood V of x contained in U.
   (a) Prove that X is locally connected if and only if for every open set U of X, each connected component of U is open in X.
   (b) Prove that if $p : X \to Y$ is a quotient map and X is locally connected, then Y is locally connected.

Part 2 - 2 hours

Try these problems. On the exam you will be asked to complete three of the four questions in 3 hours.

1. (May 2013) Let X be the space obtained from the 2-sphere S^2 by identifying the north and south poles (i.e. by identifying two diametrically opposite points).
   (a) Show that X is homotopy equivalent to $S^1 \vee S^2$.
   (b) Describe all connected covering spaces of X.
2. (May 2013)
   (a) Explain in detail how the Seifert-van Kampen theorem may be used to calculate the fundamental group of a wedge sum $X \vee Y$ of two spaces under suitable assumptions on the spaces. Clarify what assumptions on the spaces you are using and how you are using them.
   (b) Describe the presentation complex X_G of the group $G = \langle a, b, c : a^2 = 1\rangle$ as a wedge sum of familiar spaces. Explain carefully what results you are using.
3. (May 2013) Let Y be the standard 3-simplex Δ^3 with a total ordering on its four vertices. Let X be the Δ-complex obtained from Y by identifying, for each $k \leq 3$, all of its k-dimensional faces such that the identifications respect the vertex ordering. Thus X has a single k-simplex for each $k \leq 3$. Compute the simplicial homology groups of the Δ-complex X.

4. (May 2013)

   (a) Describe how to construct a cell structure on the 2-sphere S^2 consisting of one 0-cell, one 1-cell, and two 2-cells, and explain how to use this cell structure to calculate the simplicial homology groups of S^2.

   (b) Explain how a long exact sequence may be used to calculate all of the (singular) homology groups $H_i(S^n)$ of the n-sphere S^n (and calculate these groups for all i and n).

Part 3 - 5-10 minutes

Wrap-up discussion with everyone:

1. Which of these problems did you find easy? Hard?
2. Which topics in the theme do you need to review more?