BASIC EXAM: SPRING 2018

Test instructions:

Write your UCLA ID number on the upper right corner of each sheet of paper you use. **Do not write your name anywhere on the exam!!!**

All answers must be justified. If you wish to use a known theorem, make sure to give a full and precise statement.

Work out 10 problems, including at least 4 of the first 6 problems and at least 4 of the last 6 problems. Clearly indicate which 10 problems you want us to grade.

1	2	3	4
5	6	7	8
9	10	11	12

Problem 1. Prove that functions $e^t, e^{2t}, \ldots, e^{nt}$ are linearly independent in the space of continuous functions on the interval $[1, 2]$.

Problem 2. Let A and B be two real 5×5 matrices, such that $A^2 = A$, $B^2 = B$ and $1 - (A + B)$ is invertible. Prove that rank(A) =rank(B).

Problem 3. Let $A = (a_{ij})$ be a complex $n \times n$ matrix. Recall that matrix

$$e^A = 1 + A + \frac{A^2}{2!} + \frac{A^3}{3!} + \ldots$$

is well defined. Suppose $e^A = 1 + A + A^2$. Prove or disprove: A is a zero matrix.

Problem 4. Consider the following matrices:

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \qquad B = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad C = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix} \qquad E = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad F = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

Which pairs of these matrices are similar over $\mathbb{R}$? You must fully justify your answer.

Problem 5. Let A, B two positive definite 2×2 matrices. Prove or disprove:
(1) $A + B$ is also positive definite,
(2) $AB + BA$ is also positive definite.

Problem 6. Compute the determinant of the following matrix:

$$\begin{pmatrix} 1 & 2 & 4 & 8 & 16 \\ 2 & 4 & 8 & 16 & 1 \\ 4 & 8 & 16 & 1 & 2 \\ 8 & 16 & 1 & 2 & 4 \\ 16 & 1 & 2 & 4 & 8 \end{pmatrix}$$

Problem 7. Prove that for each $p \in \mathbb{N}$, the infinite series

$$\sum_{n=1}^{\infty} \frac{\sin(\pi n/p)}{n}$$

converges.

Problem 8. Consider a sequence $\{x_n\}_{n=1}^{\infty}$ defined recursively by

$$x_{n+1} = \sin(x_n), \quad x_1 = 1.$$

Prove that $\lim_{n\to\infty} \sqrt{n}\, x_n$ exists and compute its value.
Hint: Show that $\frac{1}{x_{n+1}^2} - \frac{1}{x_n^2}$ converges to a constant.

Problem 9. Let $f\colon \mathbb{R} \to \mathbb{R}$ be non-decreasing (but not necessarily continuous). Prove that f is Riemann integrable on any finite interval $(a, b) \subset \mathbb{R}$.
Note: If you choose to approach this via Lebesgue integration theory, provide complete statements *including proofs* of all facts you are using.

Problem 10. Let $n \in \mathbb{N}$ and let $U \subset \mathbb{R}^n$ be non-empty, open and connected. Suppose that $f\colon U \to \mathbb{R}$ is such that all the first partial derivatives of f (exist and) vanish at each point of U. Prove that f is constant.

Problem 11. Let (X, ρ) be a compact metric space and let $f\colon X \to X$ be an isometry (meaning that $\rho(f(x), f(y)) = \rho(x, y)$ for all $x, y \in X$). Prove that f is onto; that is, $f(X) = X$.
Hint: Consider $x \notin f(X)$ and follow the iterates of f on x.

Problem 12. Let $\mathcal{F}$ be a family of real-valued functions on a compact metric space taking values in $[-1, 1]$. Prove that if $\mathcal{F}$ is equicontinuous, then

$$g(x) = \sup\bigl\{f(x)\colon f \in \mathcal{F}\bigr\}$$

is continuous.