Ph.D. qualifying exam: algebra
June 11, 2015

Instructions. This is a three-hour exam; no books, notes, or collaboration allowed. The exam has three parts, and in each part you must solve Problem 1 and two of the remaining problems. Good luck!

I. Groups

1. Let G be a finite group with $|G| = 150$. Prove that for at least one $k \in \{5, 10, 15, 25\}$ there exists a normal subgroup $N \trianglelefteq G$ of order k.

2. Let A be an abelian group (we will write the group multiplicatively in this problem). For a prime number p, define the following subgroups of A:

$$A^p := \{a^p : a \in A\}, \qquad A[p] := \{a \in A : a^p = 1\}.$$

(a) Prove or give a counterexample: $A/A[p] \cong A^p$.

(b) Prove: If A is finite, then $A/A^p \cong A[p]$.

3. Suppose J and L are groups, and $\varphi\colon L \to \mathrm{Aut}(J)$ is a group homomorphism.

(a) Give a definition for the group $J \rtimes_\varphi L$.

(b) The identity function gives a bijection of sets $J \rtimes_\varphi L \to J \times L$. Prove that this function is a group isomorphism if and only if φ is the trivial homomorphism sending everything to the identity.

4. Let p be a prime number, and P a finite p-group. Suppose H is a proper subgroup of P. Prove that H is also a proper subgroup of $N_P(H)$, the normalizer of H in P.

II. Rings

1(a) Find a rational canonical form and a Jordan canonical form for the following 3×3 matrix with real entries:

$$\begin{pmatrix} 0 & -1 & 2 \\ 3 & -4 & 6 \\ 2 & -2 & 3 \end{pmatrix}.$$

(b) Describe the distinct isomorphism classes of Abelian groups of order 360.

2. Let D be a commutative ring with unit. A submodule N of the D-module M is said to be pure in M just in case for every $y \in N$ and $a \in D$, $ax = y$ is solvable in N if solvable in M.

(a) Show that if N is a direct summand of M then N is pure in M.

(b) Show that if N is pure in M, $z \in M$, and $\operatorname{ann}(z + N) = (d)$, then there is $w \in M$ with $z + N = w + N$ and $\operatorname{ann}(w) = (d)$.

(c) Show that if N is pure in M, M is a finitely generated torsion module, and D is a p.i.d., then N is a direct summand of M.

3. In each item, a commutative ring R and an ideal $I \subseteq R$ are given. Determine whether I is prime, maximal, both, or neither.

(a) $R = \mathbb{C}[x]$, $I = (x^2 + 1)$.

(b) $R = \mathbb{Z}[x]$, $I = (6, x)$.

(c) $R = \mathbb{C}[x, y]$, $I = (y^2 - x^3)$.

4. Determine whether the ring $\mathbb{Z}[3i]$ is a UFD.

III. Fields

1. Let K be the splitting field of $X^4 - 12$ over $\mathbb{Q}$.

(a) Give an explicit description of K and compute its degree over $\mathbb{Q}$.

(b) What is the Galois group of K over $\mathbb{Q}$? You should identify the group as well as its action on the roots of $X^4 - 12$.

2. Assume that p is prime and $X^p - a$ is irreducible in $\mathbb{Q}[X]$. Show that the Galois group of $X^p - a$ over $\mathbb{Q}$ is isomorphic to the group (with respect to composition) of all functions $f : \mathbb{Z}/p\mathbb{Z} \to \mathbb{Z}/p\mathbb{Z}$ such that for some $k, l \in \mathbb{Z}/p\mathbb{Z}$ with $k \neq 0$,

$$f(y) = ky + l \text{ for all } y \in \mathbb{Z}/p\mathbb{Z}.$$

3. Let p be prime, n be a positive integer, $f(X) = X^{p^n} - X + 1$, and K be an algebraic closure of $\mathbb{F}_p$.

(a) If $\alpha \in K$ is a root of $f(X)$ and $a \in \mathbb{F}_{p^n} \subseteq K$, show that $\alpha + a$ is a root of $f(X)$ and $\alpha^p - \alpha + 1 \in \mathbb{F}_{p^n}$.

(b) Show that $X^p - X + 1$ is irreducible in $\mathbb{F}_p[X]$.

(c) Show that $X^{2^2} - X + 1$ is irreducible in $\mathbb{F}_2[X]$.

(d) Show that if $f(X)$ is irreducible in $\mathbb{F}_p[X]$, then $n = 1$ or $n = 2 = p$.

4. Let F be a field of characteristic zero and $\overline{F}$ be an algebraic closure of F. Suppose that K and L are fields, with $F \subseteq K \subseteq L \subseteq \overline{F}$, such that $[K : F] = 2$ and L/K is a finite Galois extension. Prove that there are at most two fields $M \subseteq \overline{F}$ conjugate to L over F (remember that $M \subseteq \overline{F}$ is conjugate to L over F just in case there is an isomorphism of L onto M which is the identity on F).

Algebra Preliminary Examination
Summer 2014

There are **six** problems in this examination. You are strongly encouraged to attempt as many as possible.

1. Prove or disprove each of the following:

   (a) If $R[x]$ is a PID, then R is a field.

   (b) All cyclic subgroups of order 15 in S_8 are conjugate.

   (c) All torsion-free $\mathbf{Z}$-modules are free.

   (d) All degree two field extensions are Galois.

2. Let G be a finite group of order 84 with 28 Sylow 3-subgroups. For any subgroup H of G, let $N_G(H)$ and $Z_G(H)$ be the normalizer and the centralizer of H in G, respectively.

   (a) Show that G has a normal Sylow 7-subgroup. Let K denote this subgroup. Describe $\mathrm{Aut}(K)$, the automorphism group of K.

   (b) Show that $Z_G(K)$ is a normal subgroup of G, and that $G/Z_G(K)$ is cyclic. (Note: since K is normal in G, G acts on K by conjugation.)

   (c) Let Q be a Sylow 3-subgroup of G. Show that $N_G(Q) = Z_G(Q) = Q$.

   (d) Show that $|Z_G(K)|$ is not divisible by 3.

   (e) Prove that $Z_G(K)$ must have order 28.

3. A local ring is a commutative ring with 1 which has a unique maximal ideal.

   (a) Show that the group of units of a local ring is precisely the set of elements lying outside of M.

   (b) Let F be a field, and $|\ | : F \longrightarrow \mathbb{R}$ such that

      (i) $|x| \geq 0$ for all $x \in F$ and $|x| = 0$ if and only if $x = 0$;

      (ii) $|xy| = |x|\ |y|$, for all $x, y \in F$;

      (iii) $|x + y| \leq \max\{|x|, |y|\}$, for all $x, y \in F$.

      Show that the set $R = \{x \in F : |x| \leq 1\}$ is a local ring.

   (c) Let M be a maximal ideal of a commutative ring R with 1 and n be a positive integer. Show that R/M^n is a local ring.

4. Let R be a ring with 1, and let I be a two sided ideal of R. Let M be a left R-module. Recall that IM is the submodule

$$IM = \{a_1 m_1 + \ \cdots a_n m_n : a_i \in I, m_i \in M\}.$$

Prove that $(R/I) \otimes_R M \cong M/IM$ as R-modules. (Hint: Construct an R-balanced map from $(R/I) \times M$ to M/IM and use the universal mapping property of tensor products.)

5. Suppose $A \in M_4(\mathbf{Q})$ is a 4-by-4 matrix of multiplicative order 6. We use A to give $\mathbf{Q}^4$ the structure of a $\mathbf{Q}[x]$-module in the usual way, by setting $x \cdot v = Av$ for $v \in \mathbf{Q}^4$.

   (a) What are all of the isomorphism classes of $\mathbf{Q}[x]$-modules that can arise this way?

   (b) For each of your answers above, write down the rational canonical form of the corresponding matrix A.

   (c) What are all the of the abelian groups of size 24 (up to isomorphism)?

6. Let K be the splitting field of $x^3 + 2$ over $\mathbf{Q}$.

   (a) What is the dimension of K over $\mathbf{Q}$?

   (b) What is the Galois group of K over $\mathbf{Q}$? (Identify the isomorphism type of the group, and also indicate how its elements act as permutations of the roots of $x^3 + 2$.)

   (c) Show that K contains a primitive third root of unity ζ_3. What is the Galois group of K over $\mathbf{Q}(\zeta_3)$? What is the Galois group of $\mathbf{Q}(\zeta_3)$ over $\mathbf{Q}$? Again identify these groups as permutation groups and indicate their isomorphism type.

   (d) How are the three groups discussed in this problem related?

Wesleyan University

Department of Mathematics

PhD Qualifying Exam, Written Part: Algebra

Profs. Hovey and Scowcroft

July 10, 2013

This is a 3-hour exam; no books or notes or consultations are permitted. The exam has three parts. In each part you must do problem 1 and two of problems 2 through 4. Good luck!

1. Groups

(1) (a) State all parts of Sylow's Theorem.
 (b) Show that if $0 < p < q$ are primes and $p \nmid q-1$, then any group of order pq is cyclic.

(2) Let G be a finite group and H be a subgroup of G of index n. Show that H has a subgroup K such that K is normal in G and $[G : K]$ divides $n!$.

(3) Let p be the smallest prime dividing the order of the finite group G. Show that any subgroup of G of index p is normal in G.

(4) Present and explain an example refuting the converse to Lagrange's Theorem.

2. Rings

(1) Let R be a ring (with unit).
   (a) What is an ideal of R?
   (b) If I and J are ideals of R, prove that there is a least (with respect to inclusion) ideal of R that contains both I and J as subsets. This new ideal is called the join of I with J: $I \vee J$.
   (c) If I and J are ideals of R, let

$$IJ = \{\sum_{i=1}^{n} x_i y_i : n > 0, x_1, \ldots, x_n \in I, \text{ and } y_1, \ldots, y_n \in J\}.$$

   Show that IJ is an ideal of R that is contained in both I and J. If K is any ideal of R that is contained in both I and J, must K be contained in IJ?

(2) (a) List all abelian groups of order 60.
   (b) List all $\mathbb{Q}[x]$-modules that are annihilated by $x^8 - 1$ and have dimension 4 when thought of as vector spaces over $\mathbb{Q}$.

(3) Form an abelian group M as the quotient of $\mathbb{Z}^3$ by the elements $(0, 4, 2)$, $(-1, -4, -1)$, and $(0, 0, -2)$. What abelian group is M? (That is, find its isomorphism type).

(4) Suppose that R is a principal-ideal domain and $a, b, c, d \in R$ with $a, b \neq 0$.
   (a) Find a generator for the ideal $(a) \vee (b)$ (for the notation '$\vee$,' see the first problem in this section).
   (b) For $u, v \in R$ and I any ideal of R, write

$$u \equiv v \bmod I$$

   for

$$u - v \in I.$$

   Show that

   There is $x \in R$ with $x \equiv c \bmod (a)$ and $x \equiv d \bmod (b)$

   just in case

$$c \equiv d \bmod (a) \vee (b).$$

3. Fields

(1) Let K be the splitting field of $x^4 - 6$ over $\mathbb{Q}$.
  (a) What is the dimension of K over $\mathbb{Q}$?
  (b) What is the Galois group of K over $\mathbb{Q}$? You must not only identify the group, but describe how it acts on the roots of $x^4 - 6$.
  (c) How many subfields does K have of each possible dimension over $\mathbb{Q}$?
  (d) How many of those subfields in each dimension are Galois over $\mathbb{Q}$?

(2) (a) What finite groups G can occur as collections of automorphisms of some field F? More precisely, for which G does there exist a field F such that G is a subgroup of the automorphism group of F? Explain.
  (b) What finite groups G can occur as collections of automorphisms of some finite field F? Explain.

(3) (a) Suppose K/F is a field extension. Prove that the elements of K that are algebraic over F form a subfield of K.
  (b) Find the minimal polynomial of $2\sqrt{2} - \sqrt{3}$ over $\mathbb{Q}$.

(4) Suppose K is a field. Let $K(x)$ denote the field of rational functions in one variable x over K, and let G be the subgroup of the automorphism group of $K(x)$ over K generated by σ, where $\sigma(x) = 1 - (1/x)$. Find a specific $u \in K(x)$ such that the fixed field of G is $K(u)$, and find the minimal polynomial of x over $K(u)$.

ALGEBRA QUALIFYING EXAM, 13 JUNE 2012

1. Groups

You must do problem (1) in this section and you must do *two* of problems (2) through (4).
You may assume without proof that the alternating group A_5 is the smallest non-abelian simple group.

(1) (a) Let G be a finite group of order $p^k m$ where p is prime and $(p, m) = 1$. Let n_p be the number of subgroups of G of order p^k. State as many facts as you can about the value of n_p. (No proofs are required for your answers in this part.)
  (b) Show that a non-abelian simple group G has no subgroup of index ≤ 4.

(2) For each of the following, either give an example or explain why no such example exists. One or two sentence answers will suffice; you don't need to give formal proofs.
  (a) A non-abelian group all of whose proper subgroups are normal.
  (b) An abelian group with a subgroup which is not normal.
  (c) A non-abelian group of order p^2 where p is prime.
  (d) An abelian group G of order n and a divisor d of n such that G has no subgroup of order d.
  (e) An infinite group all of whose elements have finite order.
  (f) A prime p and a finite group G of order $p^{k+1}m$ where p does not divide m, such that G has non-isomorphic subgroups of order p^k.

(3) (a) Find Sylow subgroups P_2, P_3, and P_5 for the three primes $2, 3, 5$ dividing the order of A_5. (It suffices to find generators of each of these groups.)
  (b) Find the normalizers in A_5 of each of the three groups you found in part (a).
  (c) For each of the divisors $d = 6, 10, 15$ of 60, either find a subgroup of A_5 of order d or explain why none exists for that choice of d.

(4) Show that a group of order 30 must have a subgroup of order 15.

2. Rings

You must do *three* of the following 4 problems.
Conventions: All rings are rings with 1, and an integral domain is commutative.

(1) For each of the following, either give an example or explain why no such example exists. One or two sentence answers will suffice; you don't need to give formal proofs.
  (a) A commutative ring R and an ideal I of R that is prime but not maximal.
  (b) A commutative ring R and a finitely generated, torsion-free R-module M that is not free.
  (c) A PID R and an R-module that is torsion-free but not free.
  (d) A Euclidean domain which is not a UFD.
  (e) A UFD which is not a PID.
  (f) An integral domain which is not Noetherian.

(2) (a) State the structure theorem for finitely generated modules over a PID. You may state either the invariant factors or elementary divisors version.
  (b) Determine all abelian groups of size 72 up to isomorphism. Explain how this relates to part (a).
  (c) Determine all conjugacy classes of 3x3 matrices over Q with characteristic polynomial $x^3 - 2x^2 + x$. Explain how this relates to part (a).

(3) Suppose that R is a commutative ring and I and J are ideals of R. Suppose that $I + J = R$.
  (a) Prove that $IJ = I \cap J$.
  (b) Prove that $R/(IJ)$ is isomorphic to $R/I \oplus R/J$. (This is the Chinese Remainder Theorem).

(4) Consider the polynomial ring $Z[x]$. Let $I = (2, x^2 + 3x - 1)$ be the ideal of $Z[x]$ generated by 2 and $x^2 + 3x - 1$. Describe the structure of the quotient ring $Z[x]/I$ in as much detail as you can. Do the same for the quotient ring $Z[x]/J$ where $J = (2, x^3 + 3x^2 - x + 4)$.

3. Fields

You must do problems (1) and (2) in this section and you must do *one* of problems (3) and (4).

(1) Let K be the splitting field of $x^3 - 5$ over Q.
   (a) What is the dimension of K over Q?
   (b) What is the Galois group of K over Q? (Identify the isomorphism type of the Galois group, and also indicate how its elements act as permutations of the roots of f.)
   (c) What are all the subfields of K? Be sure to explain how you know you have included all of them.
   (d) Which of these subfields are Galois over Q?

(2) Give an example of each of the following or explain why no such examples exist. One or two sentence answers will suffice; you don't need to give formal proofs.
   (a) A Galois extension K/F and an intermediate field E ($F \subset E \subset K$) such that F/E is not Galois.
   (b) A Galois extension K/F and an intermediate field E ($F \subset E \subset K$) such that E/K is not Galois.
   (c) A finite field of size 8.
   (d) A subfield E of the splitting field of $x^5 + x - 1$ with $[E : \mathrm{Q}] = 7$.
   (e) A field extension K/F which is finite dimensional but not separable.
   (f) An irreducible fifth degree polynomial over Q which is solvable by radicals.

(3) (a) Prove that $K = (\mathrm{Z}/2\mathrm{Z})[y]/(y^4 + y^3 + y^2 + y + 1)$ is a field.
   (b) Prove that $x^4 + x + 1$ and $x^2 + x + 1$ factor completely over the field K above.
   (c) Prove that $x^3 + x + 1$ is irreducible over the field K above.

(4) Suppose K/F is a Galois extension and that $\mathrm{Gal}(K/F)$ is isomorphic to A_n, with $n \geq 5$. Suppose $\alpha \in K$, Prove that the minimal polynomial of α over K has degree at least n. (You may use whatever group theoretic facts you know about A_n without proof.)

$\alpha \notin F$.

Algebra Preliminary Examination

Summer 2011

Do as much as you can. You should try to say something about every problem.

1. For each of the following statements, determine if it is true or false. Explain your answers.
   (a) The field $\mathbb{Q}(\sqrt[3]{2})$ is contained in some cyclotomic extension of $\mathbb{Q}$.
   (b) Let $\mathbb{F}$ be a finite field, and $f(x) \in \mathbb{F}[x]$ be an irreducible polynomial of degree n. Then the splitting field of $f(x)$ over $\mathbb{F}$ has degree n over $\mathbb{F}$.
   (c) There is a surjective group homomorphism $\varphi : S_4 \longrightarrow A_3$.
   (d) There is a surjective group homomorphism $\psi : S_4 \longrightarrow S_3$.
   (e) There is a transitive group action of S_4 on a set of three elements.

2. Let R be a commutative ring with 1. Suppose that every principal ideal of R is prime. Prove that R is a field.

3. For $n = 3, 4$, and 5, determine the number of similarity classes of matrices $A \in M_n(\mathbb{Q})$ such that $A^8 = I$ but $A^4 \neq I$.

4. (a) Prove that every group of order 15 is cyclic.
   (b) Prove that every group of order $255 = 3 \cdot 5 \cdot 17$ is cyclic.

5. Let R be a PID with field of fractions F. Let V be a vector space of dimension n over F.
   (a) Show that every finitely generated R-module in V is free of rank $\leq n$.
   (b) Let M and N be free R-modules of rank n in V. Show that there exists a nonzero $\alpha \in R$ such that $\alpha M \subseteq N$. Use this to show that there exists a basis $\{e_1, \ldots, e_n\}$ of M over R and nonzero elements $\beta_1, \ldots, \beta_n$ in F such that $\{\beta_1 e_1, \ldots, \beta_n e_n\}$ is a basis of N over R.

6. Let $p < q$ be two distinct odd primes such that $p \nmid q - 1$. Let $n = pq$ and ζ_n be a primitive n-th root of unity in $\mathbb{C}$.
   (a) Show that the splitting field E of $(x^p - 2)(x^q - 2)$ over $\mathbb{Q}$ is $\mathbb{Q}(\zeta_n, \sqrt[n]{2})$. (Note: $\gcd(p, q) = 1$; so there are integers a and b such that $pa + qb = 1$.)
   (b) Determine $[E : \mathbb{Q}]$.
   (c) Find two subgroups of $\mathrm{Gal}(E/\mathbb{Q})$ of order $(p-1)(q-1)$.
   (d) Show that the Sylow p-subgroup of $\mathrm{Gal}(E/\mathbb{Q})$ is normal and determine its fixed field.

Algebra Qualifying Exam: June 16, 2010

There are three sections of problems: in each section, please do Problem 1, together with two of the other three problems. Indicate clearly which problem in each section you choose to omit. Unless otherwise stated, you must provide justification for each of your answers. Good luck!

Groups:

1. State the Sylow Theorems. Include information about the existence, order, and number n_p of Sylow p-subgroups of a finite group, and mention additional divisibility and congruence properties of n_p.

2. Let A be a finite abelian group of order n. Prove that A is cyclic if and only if A has a unique subgroup of each order k such that k divides n.

3. (Frattini) Let K be a normal subgroup of a finite group G, and let P be a Sylow p-subgroup of K. Recall $N_G(P)$ denotes the normalizer of P in G. Show that $G = KN_G(P)$.

4. For each of the three orders $n = 15, 20, 21$, either find a non-abelian group of order n or prove that all groups of order n are abelian.

Rings and Modules:

1. Give an example of each of the following. Justification is not necessary.

   (a) A principal ideal domain D that is not a field.

   (b) A unique factorization domain D that is not a principal ideal domain.

   (c) A commutative ring R, and a torsion-free R-module M, such that M is not a free R-module.

   (d) A commutative ring R, and two non-trivial R-modules M and N such that $M \otimes_R N \cong \{0\}$.

2. Let R be a commutative ring with 1. We say R is a *local* ring if R has exactly one maximal ideal, M. Prove that, in a local ring R, any $r \in R$ is either a unit or an element of the maximal ideal M.

3. Let A be a $\mathbb{Z}$-module, and suppose $m \in \mathbb{Z}^+$. Prove that $A/mA \cong A \otimes_{\mathbb{Z}} (\mathbb{Z}/m\mathbb{Z})$.

4. Let A be the following matrix (over $\mathbb{Q}$)

$$\begin{bmatrix} -2 & 1 & 1 \\ -4 & 3 & 1 \\ -6 & 6 & -1 \end{bmatrix} \tag{1}$$

   Find the rational canonical form for A. Give a 3×3 matrix B which has the same characteristic polynomial as A, but which is not similar to A.

Fields and Galois Theory:

1. Suppose the field K is a Galois extension of the field F, such that the Galois group of K over F is isomorphic to the 8 element dihedral group D_8.

   (a) Determine the number of fields L such that $F \subset L \subset K$, $L \neq F, L \neq K$, and indicate the dimension of L over F in each case.

   (b) Refine your answer to (a) by indicating how many fields in each dimension are Galois over F.

2. Give an example of each of the following:

   (a) Fields $F \subset K \subset L$ such that L/K and K/F are Galois, L/F is algebraic, but L/F is not Galois.

   (b) A pair of fields $F \subset K$ such that K is finite dimensional over F but K is not separable over F.

3. Let $\mathbb{F}_p$ be the finite field with p elements, and let $f(x) \in \mathbb{F}_p[x]$ be a polynomial of degree 4.

   (a) Show that $f(x)$ is irreducible if and only if the polynomials $f(x)$ and $x^{p^2} - x$ are relatively prime.

   (b) Assuming $f(x)$ is irreducible over $\mathbb{F}_p$, describe the Galois group of $f(x)$ over $\mathbb{F}_p$. Include a generating set for the Galois group in your answer.

4. Decide whether or not each of the following fields is a splitting field over $\mathbb{Q}$.

   (a) $\mathbb{Q}(\sqrt[3]{2})$.

   (b) $\mathbb{Q}(\sqrt{2} + \sqrt{5})$.