Real Analysis section of the Analysis Qualifier

Wesleyan University
Summer, 2007

1 Part I

Give precise statements of the following.

1. The dominated convergence theorem.
2. The definition of convergence in measure for a sequence of functions.
3. The definition of almost uniform convergence for a sequence of functions.
4. Egorov's theorem.
5. The definition of absolute continuity for one measure with respect to another.
6. The Radon-Nikodym theorem.

2 Part II

Do as many of these six problems as you can. Partial answers are welcome.

1. Suppose that (X, A, μ) is a measure space and $\{A_n\}_{n=1}^{\infty}$ is an increasing sequence of sets in A. That is, for each n, $A_n \subset A_{n+1}$. Prove that

$$\mu\left(\bigcup_{n=1}^{\infty} A_n\right) = \lim_{n\to\infty} \mu(A_n).$$

   Give an example to show that the corresponding statement for decreasing sequences of sets is not true in general.

2. Suppose that (X, A, μ) is a measure space and $\{f_n\}_{n=1}^{\infty}$ is a sequence of functions in $L^1(X, A, \mu)$ and f is an element of $L^1(X, A, \mu)$ such that

$$\lim_{n\to\infty} \int_X |f_n - f|\, d\mu = 0.$$

   Prove that for all $a > 0$

$$\lim_{n\to\infty} \mu\{x \mid |f_n(x) - f(x)| > a\} = 0.$$

3. Prove that if $A \subset \mathbb{R}^2$ is a Lebesgue measurable set with $m(A) > 0$, then for each $t \in (0, m(A))$, there is a subset $B \subset A$ with $\lambda(B) = t$.
   m

4. Suppose that μ is a finite Borel measure on $\mathbb{R}$ such that for some number $t > 0$ and all Borel sets A and, $\mu(A+t) = \mu(A)$. Prove that μ is the zero measure.

5. Calculate the following limit:

$$\lim_{n\to\infty} \int_{[0,1]} \frac{1+nx^2}{(1+x^2)^n} dm$$

where m denotes Lebesgue measure on $[0,1]$.

6. Let $C([0,1])$ denote the space of real-valued continuous functions on $[0,1]$, regarded as a real vector space under pointwise addition and scalar multiplication. For each $f \in C([0,1])$ we set

$$\|f\| = \int_{[0,1]} f dm$$

where m denotes Lebesgue measure on $[0,1]$.

   (a) Explain why this defines a norm on $C[0,1]$.

   (b) Is $C([0,1])$ complete with respect to this norm?

Name: ________________________________

COMPLEX ANALYSIS SECTION OF THE ANALYSIS QUALIFIER

WESLEYAN UNIVERSITY
SUMMER 2007

INSTRUCTIONS

1. This exam consists of two sections. You **must** answer all of the questions (short answer/definition) in the first section and the second section. **Choose** one question to answer from the third section.

2. Clearly indicate the work you wish to be graded. **Show all work.**

1. Definition/ Short Answer

INSTRUCTIONS: Answer **all** of the following questions. Be as complete as possible, and show all relevant work.

(1) Assume $U \subseteq \mathbb{C}$ is open. Define what it means for $f : U \to \mathbb{C}$ to be *analytic.*

(2) Let $f : U \to \mathbb{C}$ be an analytic function, and let $D \subset f(U)$ be a domain. Define what it means for g to be a *branch* of f^{-1} in D.

(3) State the *(local) Cauchy Theorem.*

(4) Define what it means for a function $f : U \to \mathbb{C}$ to be *uniformly continuous* on an open set U.

(5) Let $\gamma(t) = te^{it}$ for $0 \le t \le \pi$. Show that

$$|\int_\gamma z\, dz| \le \frac{\pi^2}{2} + \frac{\pi^3}{3}.$$

(6) Find the *winding number* of the curve

$$\gamma(t) = 4e^{it} + 2i,\ t \in [0, 2\pi]$$

about the point $z = 10 + 10i$.

(7) State the *Maximum Principle* for an analytic function f on a domain D.

(8) Define the *arc length* of a path $\gamma : [a, b] \to \mathbb{C}$.

2. Section II

INSTRUCTIONS: Answer the following questions. Be as complete as possible, and show all relevant work.

(1) Suppose an entire function $f = u+iv$ has the property that $u_x v_y - u_y v_x = 1$. Show that $f(z) = az+b$, where a is a complex number of modulus one. **Carefully** explain your reasoning.

(2) Calculate $\int_\gamma \frac{\sin z}{z^3+z} dz$, where $\gamma(t) = 2e^{it},\ t \in [0, 4\pi]$. **Show** all work.

3. Section III

INSTRUCTIONS: Answer one of the following questions. Be as complete as possible, and show all relevant work. **Clearly indicate** the problem which is to be graded.

(1) Let $f(z) = \dfrac{z-a}{1-\bar{a}z}$ be defined on $D = \{z : |z| < 1\}$, where $a \in \mathbb{C} : |a| < 1$.

Using the Maximum Principle show that f maps D onto D. **Carefully** explain your reasoning.

(2) Let f be an analytic function on the open unit disk $D = \{|z| < 1\}$ so that $f(0) = f'(0) = 0$, and so that $|f'(z)| < 1$ for all $z \in D$.

Show that $|f(z)| \leq \dfrac{|z|^2}{2}$ for all $z \in D$. **Carefully** explain your reasoning.

Hint: Represent $f(z)$ as a path integral (choose an easy path.)

(3) Let $f(z) = u + iv$ be analytic in a **general open set** U. Suppose that we also know that $v = u^2$. Answer the following question:

Question: *Can we conclude that f is constant on U?*

(a) If you believe one can conclude this, then **prove** it. **Carefully** explain your reasoning.

(b) If not, provide a **counter-example** and **modify** the statement so that f would indeed be constant under the conditions that f is analytic and $v = u^2$. **Prove** that the modified statement is true.

Complex Analysis Preliminary Examination

August 9, 2006

1 Theorems and Definitions

In the following questions you are asked to give precise definitions or precise statements of theorems. Explain every symbol you use in the context of your statement.

1. Define the *winding number* of a path γ about a point $z \in \mathbb{C}$.
2. Define the *multiplicity* of an analytic function f at a point z_0 in the domain of f.
3. State the *Maximum Principle.*
4. State the *Theorem of Casorati-Weierstrass.*
5. State *Cauchy's Integral Formula.*
6. State the *Open Mapping Theorem.*

2 Problems

Solve the following problems. **Explain your reasoning and show all work.**

1. Find and classify all of the isolated singularities of the following functions. Give full explanations, show all work!

   (a) $f_1(z) = \dfrac{\cos z - 1}{z^2}$.

   (b) $f_2(z) = \dfrac{e^{(1/z^2)}}{z-1}$.

   (c) $f_3(z) = \sqrt{z}$.

2. Show that $f(z) = x^2 + iy^2$ is differentiable at all points on the line $y = x$, but that f is nowhere analytic.
3. Suppose f and g are both analytic in a domain D whose closure is compact in $\mathbb{C}$. Show that $|f(z)| + |g(z)|$ takes its maximum on the boundary. *Hint: Consider $f(z)e^{i\alpha} + g(z)e^{i\beta}$ for appropriate α and β.*
4. Let f be a non-constant entire function. Prove that $f(\mathbb{C})$ is dense in $\mathbb{C}$.
5. Find the value of the integral $\displaystyle\int_{|z|=1} \frac{\mathrm{Log}(z+e)}{z^2} dz$. Show all work!

REAL ANALYSIS PRELIMINARY EXAMINATION, 2006

1. Let X be a metric space with metric d.

a. Give the definition of a Cauchy sequence in (X, d).

b. Prove that every convergent sequence in (X, d) is a Cauchy sequence.

c. Define what it means for (X, d) to be complete.

d. State without proof one of the forms of Baire's category theorem.

2. Let N be a linear space (i.e. a vector space) over the scalar field of real numbers.

a. Give the definition of a norm on N.

b. If N is also the vector space of real numbers, and if $0 < p < \infty$, determine for which p the equations

$$||x||_p := |x|^p$$

define a norm on N.

c. These normed spaces are Banach spaces. Define in general what it means for a normed space to be a Banach space.

d. State without proof the Hahn-Banach theorem for a general normed linear space.

e. Let B and B' be Banach spaces, and let $T : B \to B'$ be a one-to-one continuous linear transformation from B onto B'. Further, let x_n be a sequence of elements of B and set $y_n = T(x_n)$. If the sequence y_n in B' is a Cauchy sequence, what can you say about the nature of the sequence x_n in B, and why?

3. Let Ω be a nonempty set and let $\mathcal{A}$ be a σ-algebra of subsets of Ω.

a. If $A_1, A_2, \cdots \in \mathcal{A}$, show that the subset $\overline{A}$ of all points $\omega \in \Omega$ which belong to infinitely many of the sets A_n is also an element of $\mathcal{A}$.

b. Let μ be a finite measure on the measurable space $(\Omega, \mathcal{A})$, and suppose that

$$\sum_{n=1}^{\infty} \mu(A_n) = 55.$$

Determine the value of $\mu(\overline{A})$.

c. For each $\omega \in \Omega$, denote by $f(\omega)$ the number of different sets A_n to which ω belongs; if $\omega \in \overline{A}$ then this number is infinity. Show, using course theorems, that this extended real-valued function is $\mathcal{A}$-measurable and that under the condition in b. above, f is μ-integrable.

d. For normalized Lebesgue measure on the unit interval, calculate the measure of the set of all numbers in the unit interval for which the first even digit in their decimal expansion is either a 4 or an 8.

REAL ANALYSIS PRELIMINARY EXAMINATION, 2006

1. Let X be a metric space with metric d.

a. Give the definition of a Cauchy sequence in (X, d).

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c. Define what it means for (X, d) to be complete.

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a. Give the definition of a norm on N.

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b. Let μ be a finite measure on the measurable space $(\Omega, \mathcal{A})$, and suppose that

$$\sum_{n=1}^{\infty} \mu(A_n) = 55.$$

Determine the value of $\mu(\overline{A})$.

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d. For normalized Lebesgue measure on the unit interval, calculate the measure of the set of all numbers in the unit interval for which the first even digit in their decimal expansion is either a 4 or an 8.

Name: ____________________________

COMPLEX ANALYSIS SECTION OF THE ANALYSIS QUALIFIER

WESLEYAN UNIVERSITY
SUMMER 2005

INSTRUCTIONS

1. This exam consists of two sections. You **must** answer all of the questions (short answer/definition) in the first section, and **choose three** questions to answer from the second section.

2. Clearly indicate the work you wish to be graded. **Show all work.**

1. Definition/ Short Answer

INSTRUCTIONS: Answer **all** of the following questions. Be as complete as possible, and show all relevant work.

(1) Assume $U \subseteq \mathbb{C}$ is open. Define what it means for $f : U \to \mathbb{C}$ to be *analytic.*

(2) Give the definition for a *cycle.*

(3) State the *(global) Cauchy Theorem.*

(4) Define what it means for a function $f : U \to \mathbb{C}$ to be *uniformly continuous* on an open set U.

(5) Calculate $h'(z)$ for $h(z) = \left(\frac{z^2 - 1}{e^z}\right)^{10}$.

(6) Calculate the *winding number* of the curve

$$\gamma(t) = 4e^{it} + 2i, \ t \in [0, 2\pi]$$

about the point $z = 10 + 10i$.

(7) State *Liouville's Theorem.*

(8) Calculate $\int_{|z|=2} \frac{e^{\pi z}}{(z-1)^2} dz$.

2. PROBLEM SECTION

INSTRUCTIONS: Answer **three (3)** of the following questions. Be as complete as possible, and show all relevant work. **Clearly indicate** which problems are to be graded.

(1) Expand the function $f(z) = e^{1/z^2}$ in a Laurent series in the annulus $A = \{z : 0 < |z| < \infty\}$. **Show all work.**

(2) Let $f(z) = a_0 + a_1 z + \ldots a_n z^n$, where n is an integer greater than or equal to one and $a_i \in \mathbb{C}$, with $a_n \neq 0$.

Prove: There exists at least one point $z \in \mathbb{C}$ so that $f(z) = 0$. (Hint: Louiville's theorem.)

(3) Let $f(z) = e^{z^2}$, let S be the square

$$S = \{z : 0 \leq x \leq 2 \text{ and } 0 \leq y \leq 2\}.$$

(a) Compute the *maximum value* of $|f(z)|$ on the square S.

(b) Find the *set of points* in the square S that realizes this maximum.

(4) Let $f(z)$ be entire. Assume for all $z \in \mathbb{C}$ that $|f(z)| \leq M|z|^n$ for a constant M and a fixed positive integer n.

Prove: f is a polynomial of degree less than or equal to n.

(5) Suppose that f is analytic in the open unit disk D and that $|f(z)| < 1$. Consider the analytic function $g : D \to D$ defined by

$$g(z) = \frac{f(z) - a}{1 - \overline{a}f(z)}$$

where $a = f(0)$.

Prove using the function $g(z)$ that $f(z)$ satisfies the inequality

$$|f(z)| \leq \frac{|f(0)| + |z|}{1 - |f(0)||z|}$$

for all $z \in D$.

(**Hint:** You will need the fact (given to you!) that g maps D to D.)

REAL ANALYSIS PRELIMINARY EXAMINATION, 2005

1. Let X be a nonempty set, and let $\mathcal{A}$ be a collection of subsets of X.

a. Define what it means for $\mathcal{A}$ to be a σ-algebra.

b. Using your definition, prove that if $A_1, A_2, \cdots$ is a sequence of subsets of X, each belonging to $\mathcal{A}$, then the subset $\overline{A}$ of all elements of X which belong to infinitely many sets A_i of the sequence, also belongs to $\mathcal{A}$.

c. Prove also that the subset $\underline{A}$ of those points of X which belong to all but finitely many of the sets A_i of the sequence, also belongs to $\mathcal{A}$.

d. Discuss the relationship between the sets $\underline{A}$ and $\overline{A}$.

e. If X contains exactly five points, determine the possible cardinalities for σ-algebras on X.

2.

Let $X = [0, 1]$, let $\mathcal{A}$ be the σ-algebra of Borel subsets of X, and denote by λ normalized Lebesgue measure on $(X, \mathcal{A})$.

a. Calculate the measure $\lambda(A)$ for each of the following sets A. (You do not need to show that the sets are measurable, but you should give brief reasons for your answers, not simply numbers.)

i) $[1/3, 2/5]$
ii) $\mathbf{Q} \cap [0, 1]$, where $\mathbf{Q}$ is the set of rational numbers
iii) $\mathbf{I} \cap [0, 1]$, where $\mathbf{I}$ is the set of irrational numbers
iv) $\mathbf{E} \cap [0, 1]$, where $\mathbf{E}$ is the set of algebraic numbers

b. State without proof the monotone convergence theorem for the measure space $(X, \mathcal{A}, \lambda)$.

c. Recall the standard Cantor procedure on the closed unit interval: At the zeroth stage, the interval $(1/3, 2/3)$ is removed from $[0, 1]$, leaving two intervals $[0, 1/3]$ and $[2/3, 1]$. At the first stage, the middle thirds of each of these intervals are removed, leaving four intervals, and, continuing in this way, at the nth stage, the middle thirds of the remaining 2^n intervals are removed, leaving 2^{n+1} intervals. The set which remains after countably many steps, which we denote by C, is the Cantor set. Now define the function

$$f : [0, 1] \longrightarrow [0, \infty)$$

by requiring that $f(x)$ be equal to $\frac{1}{2^n}$ if the point x belongs to an interval removed at the nth stage, and that $f(x)$ be equal to π if x belongs to the Cantor set C.

Prove that f is measurable (you may use any theorem you recall from the course) and calculate

$$\int f d\lambda.$$

3. Let B be a real vector space.

a. Give the definition of a *norm* on B.

b. Recall that a norm on B can be used to define a metric on B; give the corresponding formula for this metric, and prove that it is a metric, using your definition in the previous question.

c. Say what it means for B to be a Banach space.

d. State the uniform boundedness theorem for a collection of bounded linear transformations from the Banach space B to a normed linear space N, and succinctly explain the main idea in the proof of this theorem. (This theorem is also known as the Banach-Steinhaus theorem.)

Preliminary Examination
Analysis
August 2, 2004

This exam is in two parts, Complex Analysis and Real Analysis. You are expected to demonstrate mastery in both parts. In particular, you are expected to do at least three of the problems in sections 1.3 and at least three of the problems in section 2.3.

1 Complex Analysis

1.1 Theorems and Definitions

(5 points each)

In the following questions you are asked to give precise definitions or precise statements of theorems. Explain every symbol you use in the context of your statement.

1. State the Theorem of Casorati-Weierstrass.
2. State the Residue Theorem.
3. State the most general version of Cauchy's Theorem.
4. State the Open Mapping Theorem.

1.2 Examples

(5 points each)

Give examples of the following.

1. Give an example each of an analytic function that has a removable singularity, a pole, and an essential singularity at the origin.
2. The function $f : \mathbb{C} \to \mathbb{C}$, given by

$$f(z) = \begin{cases} e^{-\frac{1}{z^4}}, & \text{if } z \neq 0; \\ 0, & \text{if } z = 0. \end{cases}$$

   satisfies the Cauchy-Riemann equations at every point of the complex plane (you don't have to verify this). In spite of this, f is not an analytic function. How is this possible?

1.3 Problems

(20 points each)

Solve the following problems. Explain your reasoning and show all work.

1. Let f be an entire function so that $\mathrm{Re} f(z) \le 10$ for all $z \in \mathbb{C}$. Show that f is a constant function.

2. Suppose that D_1 and D_2 are simply connected domains in the complex plane whose intersection is non-empty and connected. Prove that $D_1 \cap D_2$ and $D_1 \cup D_2$ are simply connected.

3. If $\gamma = \beta + [4\pi, 0]$, where $\beta(t) = te^{it}$ for $0 \le t \le 4\pi$, evaluate the integral
$$\int_\gamma \frac{z^2+1}{(z+1)(z+4)} dz.$$

4. Suppose that $f = u + iv$ is analytic in a domain $D \subset \mathbb{C}$, and that $v(z) = (u(z))^2$ holds for every $z \in D$. Show that f is constant in D.

   Hint: Differentiate the relation $u^2 - v = 0$ with respect to x and y.

5. Let D be a bounded domain in the complex plane. Suppose that every function in a sequence $\{f_n\}$ is continuous on $\overline{D}$ and analytic in D. Given that this sequence converges uniformly on ∂D, prove that it converges uniformly on D.

6. If f and g are entire functions and if $|g(z)| \le |f(z)|$ for every $z \in \mathbb{C}$, show that $g(z) = cf(z)$ for all $z \in \mathbb{C}$ for some fixed constant c.

7. Let f have a pole of order m at a point z_0, and let g have a pole of order n at the same point. Show that $f + g$ has either a removable singularity at z_0 or a pole of order not greater than $\max\{m, n\}$ at z_0.

2 Real Analysis

2.1 Statements of Theorems

(5 points each)

1. State the Lebesgue monotone convergence theorem.

2. State the Radon-Nikodym theorem.

3. State a theorem which describes how Lebesgue measurable functions on $[0, 1]$ can be approximated by continuous functions.

2.2 Examples

(5 points each)

In the following section, the required examples may be presented without proof that they have the desired properties.

1. Give an example of a subset of $[0,1]$ with positive Lebesgue measure, whose closure contains no non-empty open intervals.
2. Give an example of a continuous function of bounded variation that is not absolutely continuous.
3. Let m denote Lebesgue measure on $\mathbb{R}$.
   (a) Give an example of a real-valued function f on $\mathbb{R}$ which is in $L^2(\mathbb{R}, m)$ but not in $L^3(\mathbb{R}, m)$.
   (b) Give an example of a real-valued function g on $\mathbb{R}$ which is in $L^3(\mathbb{R}, m)$ but not in $L^2(\mathbb{R}, m)$.

2.3 Problems

(20 points each)

Solve the following problems. Explain your reasoning and show all work.

1. Suppose that f and g are real-valued measurable functions on a measurable space $(X, \mathcal{B})$. Prove that their sum $f+g$ is measurable.
2. Suppose that $\{f_n\}_n$ is a sequence of functions converging pointwise to f on the interval $[a,b]$. Prove that
$$T_a^b(f) \leq \underline{\lim}_n T_a^b(f_n).$$
3. Suppose that $g : [0,1] \to \mathbb{R}$ is absolutely continuous. Let $E \subset [0,1]$ be a set of Lebesgue measure zero. Prove that $g(E)$ is a set of Lebesgue measure zero.
4. Suppose that $f : [0,1] \to [0,1]$ is a Lebesgue measurable function. Prove that the graph of f is a set of measure zero with respect to the completed product measure $m \times m$ on $[0,1] \times [0,1]$. (Here m denotes Lebesgue measure on $[0,1]$).
5. Suppose that f and g are integrable functions on a measure space $(X, \mathcal{B}, \mu)$ and for all sets $A \in \mathcal{B}$
$$\int_A f d\mu = \int_A g d\mu.$$
Prove that $f(x) = g(x)$ $\mu - a.e.$
6. Prove that $L^\infty(X, \mathcal{B}, \mu)$ is complete in the L^∞ norm.

PRELIMINARY EXAM
Analysis
August 4, 2003, 9:00-11:00am

I. Short Answer Questions

Instructions: The following questions ask for a definition or a short answer. The short answers should include a brief explanation or counterexample, as appropriate. A complete proof is not required.

Throughout this exam, a pair $(Y, \tau(Y))$ represents a set Y and a σ-algebra $\tau(Y)$ of subsets of Y.

1. Let $Y = \mathbb{R}$. Define the Borel σ-algebra on Y.
2. Given a pair $(Y, \tau(Y))$ as above, define what is meant by a measure on $(Y, \tau(Y))$.
3. Is there a compact subset of $[0,1] \setminus \mathbb{Q}$ with positive Lebesgue measure?
4. State the Lebesgue Dominated Convergence Theorem.
5. Let E be a Lebesgue measurable subset of $\mathbb{R}$ with $m(E) < \infty$. (Here m denotes Lebesgue measure on $\mathbb{R}$). Let $\{f_n\}_{n=1}^{\infty}$ be a sequence of real-valued measurable functions that are uniformly bounded, such that
$$\lim_{n \to \infty} f_n(x) = f(x)$$
for all $x \in E$.
Is there relationship between $\int_E f_n dm$ and $\int_E f dm$? If so, what is it?
6. Is every bounded real-valued function on $[0,1]$ which is Lebesgue integrable also Riemann integrable?
7. Define what it means for a measure μ to be absolutely continuous with respect to a measure ν on $(X, \tau(X))$.
8. Let μ be a measure defined on $(Y, \tau(Y))$, and let F_n be a decreasing sequence of sets in $\tau(Y)$.

   Is
$$\mu\left(\bigcap_{n=1}^{\infty} F_n\right) = \lim_{n \to \infty} \mu(F_n)?$$
9. Let $\{f_n\}_{n=1}^{\infty}$ be a sequence of real-valued functions on $\mathbb{R}$. Is it true that $f_n \to 0$ uniformly on $\mathbb{R}$ implies $\int_{\mathbb{R}} f_n dm = 0$? (Here again m denotes Lebesgue measure on $\mathbb{R}$).
10. What is meant by the L^p norm of a function defined on a measure space $(Y, \tau(Y), \mu)$?

II. Problems

Instructions: Please answer as many of the following questions as you can. Supply as many details as you can. You are encouraged to give a partial answer if you are unable to answer a question completely.

1. Let $f \geq 0$, and let $(\mathbb{R}, \tau(\mathbb{R}), m)$ be the Lebesgue measure space on $\mathbb{R}$.

   Prove: the set function $A \rightarrow \int_A f dm$ is a measure.

2. Let f be a Lebesgue integrable function on $\mathbb{R}$. For each $n \in \mathbb{N}$, let $E_n = \{x \in \mathbb{R} \mid |f(x)| \geq n\}$. Show that

$$\lim_{n \to \infty} nm(E_n) = 0.$$

   (Here again m denotes Lebesgue measure).

3. Let $(X, \tau(X), \mu)$ be a measure space such that $\mu(X) < \infty$. Let $\{f_n\}_{n=1}^{\infty}$ be a sequence of real-valued measurable functions.

   Show: $f_n \rightarrow 0$ in measure if and only if

$$\int_X \frac{|f_n(x)|}{1 + |f_n(x)|} d\mu \rightarrow 0.$$

4. Let $(X, \tau(X)$ denote a set X and a σ-algebra on X. Let μ and ν be two finite measures on $(X, \tau(X))$ and assume that for each $X \in \tau(X)$ that $\mu(X) = 0$ implies $\nu(X) = 0$.

   Show: For every $\epsilon > 0$ there exists a $\delta > 0$ so that $\mu(A) < \delta$ implies $\nu(A) < \epsilon$.

5. Let $X = \mathbb{R}$, and let $B(\mathbb{R})$ be the Borel $\sigma-$algebra. Let m be the usual Lebesgue measure, and let σ be the counting measure on $\mathbb{R}$.

   Let $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ be the characteristic function of the line $y = x$. That is,

$$f(x, y) = \left\{ \begin{array}{l} 1, \text{ if } x = y \\ 0, \text{ if } x \neq y \end{array} \right\}.$$

   Show:

$$\int_X \int_X f(x, y) dm(x) d\sigma(y) \neq \int_X \int_X f(x, y) d\sigma(y) dm(x).$$