# Real Analysis Preliminary Examination

(supplemental)
August, 2013

## 1 Part I

1. State Fatou's lemma.
2. State Egoroff's theorem.

## 2 Part 2

Do as many of the following six problems as you can. While fully explained answers are requested, partial answers are also welcome.

1. Suppose that $A \subset \mathbb{R}$ is a Lebesgue measurable set such that for all rational numbers $r$,
$$A + r = A.$$
Prove that either $m(A) = 0$ or $m(\mathbb{R} \backslash A) = 0$.

   Here $m$ denotes Lebesgue measure, and $A + r$ denotes the translate of $A$ :
$$A + r = \{a + r : a \in A\}.$$

2. Suppose that $(X, \mathcal{B}, \mu)$ be a measure space such that $\mu(X) = 1$, and that $\{B_i\}_{i=1}^{N}$ is a sequence of sets in $\mathcal{B}$ such that for each $i$, $\mu(B_i) > \frac{1}{2}$. Show that there is a set $C \in \mathcal{B}$ of positive measure such that for all $x \in C$,
$$\frac{1}{N} \sum_{i=1}^{N} \chi_{B_i}(x) > \frac{1}{2}.$$

3. Suppose that $A \subset \mathbb{R}$ is a Lebesgue measurable set, and $f : A \to \mathbb{R}$ is a measurable function. Suppose further that for all compact sets $K \subset A$ we have
$$\int_K f\, dm = 0.$$
(Here $m$ denotes Lebesgue measure). Does it follow that $f(x) = 0$ *a.e.*?

4. .

(a) Give an example of a sequence of functions $\{f_n\}_{n=1}^{\infty}$ on $[0,1]$ which are Lebesgue integrable, such that $f_n(x) \to 0$ *a.e.*, but not in $L^1$.

(b) Give an example of a sequence of functions $\{f_n\}_{n=1}^{\infty}$ on $[0,1]$ which are Lebesgue integrable, such that $f_n(x) \to 0$ in $L^1$, but not *a.e.* .

5. .

(a) Suppose that $F : [0,1] \to \mathbb{R}$ is a strictly increasing function. Does this imply that $F' > 0$ a.e.?

(b) Suppose that $F : [0,1] \to \mathbb{R}$ is a function such that $F'(x) > 0$ a.e. Does this imply that $F$ is strictly increasing?

6. Suppose that $f : [0,1] \to \mathbb{R}$ is an absolutely continuous function. Prove, directly from the definitions, that $f$ is of bounded variation.

# Complex Analysis Preliminary Examination

August, 2013

## 1 Theorems and Definitions

In the following questions you are asked to give precise definitions or precise statements of theorems. **Explain every symbol you use in the context of your statement or definition.**

1. Define the *winding number* of a path $\gamma$ about a point $z \in \mathbb{C}$.
2. Define the *multiplicity* of an analytic function $f$ at a point $z_0$ in the domain of $f$.
3. State the *Theorem of Casorati-Weierstrass.*
4. State *Cauchy's Integral Formula.*
5. State the *Open Mapping Theorem.*

## 2 Problems

Solve **at least four** of the following problems. **Explain your reasoning and show all work.**

1. For each of the following functions, find and classify all isolated singularities. In the case of a pole, give the order of the pole and find the residue:

   (a) $f(z) = \dfrac{\cos z - 1}{z^2}$.

   (b) $f(z) = \dfrac{e^{(1/z^2)}}{z-1}$.

   (c) $f(z) = \dfrac{\sqrt{z}}{z-1}$.

2. Evaluate the following integrals:

   (a) $\displaystyle\int_{|z|=1} \frac{\operatorname{Log}(z+e)}{z^2}\,dz$.

   (b) $\displaystyle\int_{|z|=4} \left(\frac{\sin z}{z}\right)^{100} dz$.

   (c) If $\gamma = \beta + [4\pi, 0]$, where $\beta(t) = te^{it}$ for $0 \le t \le 4\pi$, evaluate the integral $\displaystyle\int_\gamma \frac{z^2+1}{(z+1)(z+4)}\,dz$.

3. Suppose $f$ and $g$ are both analytic in a domain $D$ whose closure is compact in $\mathbb{C}$, and suppose both functions extend continuously to the boundary. Show that $|f(z)| + |g(z)|$ takes its maximum on the boundary. *Hint: Consider $f(z)e^{i\alpha} + g(z)e^{i\beta}$ for appropriate $\alpha$ and $\beta$.*

4. Let $f$ be an analytic function in a domain $D$, and suppose that there exists a point $z_0 \in D$ such that $|f(z)| \geq |f(z_0)| > 0$ for all $z \in D$. Show that $f$ must be constant in $D$.

5. Suppose that $D_1$ and $D_2$ are simply connected domains in the complex plane whose intersection is non-empty and connected. Prove that $D_1 \cap D_2$ and $D_1 \cup D_2$ are simply connected.

6. Suppose that $f$ is an even analytic function in a disk $B(0, r)$, centered at 0, and of radius $r$. Show that the function $g : B(0, r) \to \mathbb{C}$, defined by $g(z) = f(\sqrt{z})$ is a well-defined analytic function. Express $g^{(n)}(0)$ (the $n$th derivative of $g$ at 0) in terms of the derivatives of $f$.

# Real Analysis Preliminary Examination

June, 2013

## 1 Part I

1. State the Lebesgue dominated convergence theorem.
2. State the Radon-Nikodym theorem.

## 2 Part 2

Do as many of the following six problems as you can. Partial answers are welcome.

1. Suppose that $(X, \mathcal{B}, \mu)$ is a measure space and $\{A_n\}_{n=1}^{\infty}$ is a sequence of sets in $\mathcal{B}$ such that
$$\sum_{n=1}^{\infty} \mu(A_n) < \infty.$$
Prove that
$$\mu\left(\bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} A_k\right) = 0.$$

2. (a) Give an example of a compact set $K \subset [0,1]$ which has positive Lebesgue measure, but which contains no non-empty open interval.

   (b) Prove that there exists a set $A \subset [0,1]$ such that for all intervals $I \subset [0,1]$ of positive measure, we have
   $$0 < m(A \cap I) < m(I).$$
   (Here $m$ denotes Lebesgue measure).

3. Suppose that $\mu$ is a finite Borel measure on $\mathbb{R}$ such that for some number $t > 0$ and all Borel sets $A$, $\mu(A + t) = \mu(A)$. (Here $A + t$ denotes $\{x + t : x \in A\}$).

   Prove that $\mu(\mathbb{R}) = 0$.

4. Suppose that $(X, \mathcal{B}, \mu)$ is a measure space and that $f$ is a non-negative integrable function on $X$.

   (a) Prove that, for each $a > 0$, $\mu\{x \in X : f(x) \geq a\} < \infty$.

   (b) Prove that $\{x \in X : f(x) > 0\}$ is a countable union of sets of finite measure.

5. Suppose that $(X, \mathcal{B}, \mu)$ is a measure space such that $\mu(X) < \infty$, and let $\{f_n\}_{n=1}^{\infty}$ be a bounded sequence of measurable functions on $X$. (That is, for some $M > 0$, and for all $n \in \mathbb{N}$ and $x \in X$, $|f_n(x)| < M$). Suppose that $f_n \to 0$ in measure.

   Prove that $\int |f_n| \, d\mu \to 0$.

6. Suppose that $g : [0,1] \to \mathbb{R}$ is an absolutely continuous function. Let $E \subset [0,1]$ be a set of Lebesgue measure zero. Prove that $g(E)$ is also a set of Lebesgue measure zero. (Here $g(E)$ denotes $\{g(x) : x \in E\}$).

# Complex Analysis Preliminary Examination

June, ~~20013~~ 2013

## 1 Theorems and Definitions

In the following questions you are asked to give precise definitions or precise statements of theorems. **Explain every symbol you use in the context of your statement or definition.**

1. Define what it means for a function $f : U \to V \subset \mathbb{C}$ to be *analytic* in the open set $U \subset \mathbb{C}$.
2. State the *Cauchy Integral Formula* for an analytic function $f$ in a disk $\Delta$.
3. State the *Residue Theorem.*
4. Define what an *essential singularity* is.
5. Define what it means for a sequence of functions $\{f_n : U \to \mathbb{C}\}$ to *converge locally uniformly* to a function $f : U \to \mathbb{C}$.

## 2 Problems

Solve **at least four** of the following problems. **Explain your reasoning and show all work.**

1. For each of the following functions, classify the singularity and find the residue at the given point:
   (a) $f(z) = z^2 e^{-1/z^3}$, $z_0 = 0$.
   (b) $f(z) = \frac{z^2}{(z-1)^2}$, $z_0 = 1$.
   (c) $f(z) = \frac{\sin(z^3)}{(1-\cos(z))^3}$, $z_0 = 0$.
2. Evaluate the following integrals:
   (a) $\int_{\{|z|=2\}} \frac{e^z}{(1+z)^2} dz$
   (b) Let $\gamma(t) = 2\cos(t) + i\sin(t)$ for $0 \le t \le 2\pi$. Evaluate $\int_\gamma \frac{1}{z^2+2iz} dz$.
   (c) Suppose that $\gamma$ is a piecewise smooth closed path in $\mathbb{C}$. Verify that $n(\overline{\gamma}, \overline{z}) = -n(\gamma, z)$, where $n$ denotes the winding number, and $\overline{\phantom{x}}$ is the complex conjugate, in particular, $\overline{\gamma}$ is parameterized via $\overline{\gamma}(t) = \overline{\gamma(t)}$.

3. Let $f$ be an entire function such that the function $g(z) = \sqrt{f(z)}$ is also an entire function. Prove that $f$ must be constant.

4. Let $\{f_n\}$ and $\{g_n\}$ be sequences of functions that converge uniformly on a set $A$. Under the assumption that $\{f_n\}$ and $\{g_n\}$ are uniformly bounded (i.e. there exists a constant $M$ such that $|f_n(z)| \leq M$ and $|g_n(z)| \leq M$ for all $z \in A$ and all $n \in \mathbb{N}$), prove that the sequence $\{f_n g_n\}$ converges uniformly on $A$.

5. Let $D \subset \mathbb{C}$ be a domain and let $f, g : D \to \mathbb{C}$ be analytic. Let $z_0 \in D$ and suppose that $f(z_0) = g(z_0) = 0$. Suppose furthermore that

$$\lim_{z \to z_0} \frac{f'(z)}{g'(z)} \in \mathbb{C}$$

exists. Show that

$$\lim_{z \to z_0} \frac{f(z)}{g(z)}$$

exists as well and calculate this limit.

6. Let $f$ be a non-constant entire function. Prove that $f(\mathbb{C})$ is dense in $\mathbb{C}$.

7. Let $f$ have a pole of order $m$ at a point $z_0$, and let $g$ have a pole of order $n$ at the same point. Show that $f + g$ has either a removable singularity at $z_0$ or a pole of order not greater than $\max\{m, n\}$ at $z_0$.

COMPLEX ANALYSIS PRELIM, 2012

# 1 Basic facts:

Please give brief but informative responses here, not overly long reproductions of a textbook's presentation.

(2 points each)

1. Give three equivalent characterizations of analyticity of a complex-valued function $f$ in a domain $D \subset \mathbb{C}$. That is, complete the sentence: "$f$ is analytic in $D$ if and only if..." in three ways.
2. State a version of a theorem that expresses the $m^{th}$ derivative of an analytic function as an integral.
3. State what is meant by the following terms:
   (a) a pole of an analytic function.
   (b) the order of a pole of an analytic function
   (c) an essential singularity of an analytic function
4. State the Casorati-Weierstrass theorem about the behavior of an analytic function in the vicinity of an essential singularity.
5. State the Residue Theorem.
6. State the Argument Principle.

## 2 Problems:

Please do as many of the following problems as you can. (We recognize that time constraints make it unreasonable to expect you to do all of them!) Partial answers will be awarded partial credit.

(12 points each)

1. .

   (a) Prove that if $A$ and $B$ are the radii of convergence of $\sum_{k=0}^{\infty} a_k z^k$ and $\sum_{k=0}^{\infty} b_k z^k$, respectively, then the radius of convergence of $\sum_{k=0}^{\infty} a_k b_k z^k$ is at least $AB$.

   (b) Given an example of two series for which the above inequality is strict.

2. .

   (a) Find all possible values of the integral

   $$\int_\gamma \frac{1}{z^2+1} dz$$

   where $\gamma$ is a closed, smooth curve in $\mathbb{C} \setminus \{i, -i\}$.

   (b) Answer the above question under the assumption that $\gamma$ is a positively oriented circle in $\mathbb{C} \setminus \{i, -i\}$, and describe which circles give which of the possible values.

3. Let $D$ be a domain in $\mathbb{C}$ with $0 \in D$, and let $f$ and $g$ be analytic functions in $D$. Suppose that $f$ and $g$ both have zeros of order 7 at $z = 0$. Prove that the limits $\lim_{z \to 0} \frac{f'(z)}{g'(z)}$ and $\lim_{z \to 0} \frac{f(z)}{g(z)}$ both exist and are equal.

4. Suppose that $f$ is an entire function and that there is some $M > 0$ and some integer $n$, so that for all $z \in \mathbb{C}$,

   $$|f(z)| \leq M |z|^n$$

   Prove that $f$ is a polynomial of degree less than or equal to $n$.

5. Suppose that $D$ is a bounded domain in $C$ and $\{f_n\}_{n=1}^{\infty}$ is a sequence of analytic functions in $D$, each of which extends to a continuous function on the closure of $D$. Suppose further that the sequence $\{f_n\}$ converges uniformly on the boundary of $D$. Prove that the sequence converges uniformly on $D$.

REAL ANALYSIS PRELIM 2012

1. Let $X$ be a nonempty set, and let $\mathcal{A}$ be a collection of subsets of X.

a. Define what it means for $\mathcal{A}$ to be a $\sigma$-algebra.

b. Using your definition, prove that if $A$ and $B$ are sets belonging to $\mathcal{A}$, then the set $A \cap B$ also belongs to $\mathcal{A}$.

c. Give an example of a nonempty set $X$, a ring $\mathcal{R}$ on $X$, and a sequence $A_1, A_2, \cdots$ of subsets of $X$, each belonging to $\mathcal{R}$, such that the subset

$$\bigcap_{n=1}^{\infty} A_n$$

does not belong to $\mathcal{R}$.

d. If $X$ contains exactly four points, determine the possible cardinalities for $\sigma$-algebras on $X$, and for each cardinality, the number of different $\sigma$-algebras having that cardinality.

2. Let $X = [0,1]$, let $\mathcal{A}$ be the $\sigma$-algebra of Borel subsets of $X$, and denote by $\lambda$ Lebesgue measure on $(X, \mathcal{A})$.

a. Calculate the measure $\lambda$ of each of the following sets. (You do not need to show that the sets are measurable, but you should give brief reasons for your answers, not simply numbers.)

i) $\{x \in X : \sin \frac{2\pi}{x} \leq 0\}$
ii) $\{x \in X : x = 0.x_1x_2\cdots, \text{no digit } x_i = 7\}$
iii) $\{x \in X : x = 0.x_1x_2\cdots, \text{for some } i > 0, x_i = 5 \text{ and } x_{i+1} = 6\}$

b. State without proof the dominated convergence theorem for the measure space $(X, \mathcal{A}, \lambda)$.

c. Define the function $f$ on $X$ as follows: if $x \neq 0$, determine a positive integer $n = n(x)$ for which the distance $d(x)$ from $x$ to the point $1/n$ is minimal, and then set $f(x)$ equal to $d(x)$ if n is odd; otherwise set $f(x)$ equal to zero. Explain why $f$ is integrable (you may use theorems you recall from the course) and calculate

$$\int f d\lambda.$$

Please turn the page for problems 3 and 4.

3. Let $V$ be a real vector space.

a. Give the definition of a *norm* on $V$.

b. Recall that a norm on $V$ can be used to define a metric on $V$; give the corresponding formula for this metric, and prove that it is a metric, using your definition in the previous question.

c. Say what it means for $V$, under this norm, to be a Banach space.

d. Give an example of a normed vector space which is not a Banach space.

e. State without proof the Uniform Boundedness Theorem, also known as the Banach-Steinhaus theorem, taking care to state the correct hypotheses on the vector spaces and mappings occurring in the theorem.

f. If $X$ and $Y$ are real Banach spaces, and if $T : X \to Y$ is a continuous one-to-one linear mapping from $X$ onto $Y$, prove the the inverse $T^{-1}$ is a linear mapping. Is in this situation this inverse mapping continuous? Please give a reason for your answer.

4. Which theorem or theorems of this course seem to be the most interesting ones, and why?

GOOD LUCK!

# Complex Analysis Prelim, Summer 2011

1. The real and imaginary parts of the square roots of $2 - 4i$ can be given by simple algebraic expressions involving only square roots, sums, and differences of real numbers. Give these two expressions for each of the square roots.

2. Represent $\frac{z^4}{z^3-1}$ by partial fractions.

3. If $f$ is holomorphic on the open unit disk, then show that $g$ is also, where $g(z) := \overline{f(\overline{z})}$.

4. Prove that if $A$ and $B$ are the respective radii of convergence of the power series $\sum a_n z^n$ and $\sum b_n z^n$, then the radius of convergence of $\sum a_n b_n z^n$ is at least $AB$.

5. Give the definition of $\int_\gamma f(z)dz$, if $\gamma$ is a differentiable curve in the complex plane and $f$ is a continuous function defined on the trace of $\gamma$.

6. State Cauchy's theorem for triangles.

7. Using the simplest form of Cauchy's estimates, prove that a bounded entire function is constant.

8. How many roots of $z^4 - 6z + 3 = 0$ have absolute values between 1 and 2?

9. A form of the lemma of Schwarz states that if $f$ is holomorphic on the open unit disk, and if $f(0) = 0$ and $|f(z)| \leq 1$ whenever $|z| \leq 1$, then $|f(z)| \leq |z|$ whenever $|z| \leq 1$. Give a proof using the maximum principle.

10. Evaluate $\int_0^\infty \frac{x^2 dx}{x^4+5x^2+6}$ using the residue theorem.

Complete as many exercises as possible in the allotted time. Good luck!

Real Analysis Prelim, Summer, 2011

Solve as many of the following eight problems as you can. Partial answers are welcome. (Note: the rather large number of questions is intended to give you more choices, not to induce despair!)

Throughout this part of the exam, Lebesgue measure is understood whenever the symbol "$m$" is used.

1. Prove or give a counterexample to the following assertion: If $\{f_n\}_{n=1}^{\infty}$ is a sequence of continuous functions on $[0,1]$ with $0 \leq f_n(x) \leq 1$ for all $n$ and $x$, and if $\lim_{n\to\infty} \int_{[0,1]} f\, dm = 0$, then $f_n(x) \to 0$ almost everywhere.

2. Let $f : \mathbb{R} \to \mathbb{R}$ be a Lebesgue measurable function. Does either of the following two conditions on $f$ imply the other?

   Condition $i$ : For almost every $x \in \mathbb{R}$, $f$ is continuous at $x$.

   Condition $ii$ : There is a continuous function $g : \mathbb{R} \to \mathbb{R}$ such that $f(x) = g(x)$ almost everywhere.

   Give proofs or counterexamples, as appropriate.

   (The term "almost everywhere" refers to Lebesgue measure here.)

3. .

   (a) Suppose $A \subset \mathbb{R}$. Define the Lebesgue outer measure $m_*(A)$ of $A$.

   (b) Prove that for all seqeunces $\{A_i\}_{i=1}^{\infty}$ of subsets of $\mathbb{R}$,

$$m_*\left(\bigcup_{i=1}^{\infty} A_i\right) \leq \sum_{i=1}^{\infty} m_*(A_i)$$

4. Suppose that $\mu$ is a finite Borel measure on $\mathbb{R}^d$. For each $t > 0$ let $S_t = \{x \in \mathbb{R}^d : |x| = t\}$. Let $A = \{t \in \mathbb{R} : \mu(S_t) > 0\}$. Prove that $A$ is (at most) countable.

# Real Analysis section of the Analysis Qualifier

Wesleyan University
July 14, 2010

## 1 Part I

Give precise statements of the following.

1. The monotone convergence theorem.
2. The Radon-Nikodym theorem
3. The spectral theorem for compact symmetric operators on a Hilbert space.

## 2 Part II

Do as many of these six problems as you can. Partial answers are welcome.

1. Suppose that $(X, \mathcal{B}, \mu)$ is a measure space and that $\{A_n\}_{n=1}^{\infty}$ is a sequence of measurable sets in $(X, \mathcal{B}, \mu)$.

   (a) Prove that, if $\sum_{n=1}^{\infty} \mu(A_n) < \infty$ then the set of $x$ which are elements of infinitely many of the sets $A_n$ has measure zero.

   (b) Give an example of a measure space $(X, \mathcal{B}, \mu)$ and a sequence of measurable sets $\{A_n\}_{n=1}^{\infty}$ in $(X, \mathcal{B}, \mu)$ such that $\sum_{n=1}^{\infty} \mu(A_n) = \infty$ but still the set of $x$ which are elements of infinitely many of the sets $A_n$ has measure zero.

2. Suppose that $\mu$ is a Borel measure on $\mathbb{R}$ with the properties:
   i. For every bounded interval $I$, $\mu(I) < \infty$ and
   ii. For every rational number $r$ and every bounded interval $I$, $\mu(I) = \mu(I + r)$. (The notation $I + r$ means $\{x + r \mid x \in I\}$).

   (a) Prove that for every $x \in \mathbb{R}$, $\mu(\{x\}) = 0$

   (b) Prove that for every bounded interval $I$ and every real number $a$, $\mu(I) = \mu(I + a)$

3. .

   (a) Give an example of a closed subset $A$ of $\mathbb{R}^2$ with positive Lebesgue measure and which contains no non-empty open set.

   (b) Suppose that $A$ is a set of the sort described in part a, and that $f : \mathbb{R}^2 \to \mathbb{R}$ is a function such that $f(x) = \chi_A(x)$ a.e. Prove that there is a set of positive measure $B$ such that for all $x \in B$, $f$ is discontinuous at $x$.

4. .

(a) Suppose that $f : \mathbb{R} \to \mathbb{R}$ is a Lebesgue integrable function. For each $t \in \mathbb{R}$ let $f_t$ denote the translate of $f$ by $t$. That is, $f_t(x) = f(x-t)$. Prove that

$$\lim_{t \to 0} \int_{\mathbb{R}} |f - f_t| \, dm = 0$$

(Here $m$ denotes Lebesgue measure).

(b) Suppose that $A \subset \mathbb{R}$ is a set of finite Lebesgue measure. What does the above result say about the sets $A + t$? (As before, $A + t = \{x + t \mid x \in A\}$.)

5. Let $(X, \mathcal{B}, \mu)$ be a measure space.

(a) Show that if $\mu(X) < \infty$, then $L^4(X, \mathcal{B}, \mu) \subset L^3(X, \mathcal{B}, \mu)$.

(b) Give an example of a measure space $(X, \mathcal{B}, \mu)$ and a function $f$ such that $f \in L^4(X, \mathcal{B}, \mu)$ but $f \notin L^3(X, \mathcal{B}, \mu)$.

6. Let $H = L^2([0,1], m)$, where $m$ denotes Lebesgue measure. We know that $H$ is a Hilbert space, with respect to the inner product $(f, g) = \int f\bar{g}dm$.

(a) Let $E$ be a Lebesgue measurable subset of $[0,1]$. Verify that $H_E = \{f \in H \mid f = f\chi_E\}$ is a closed subspace of $H$.

(b) Identify the orthogonal complement of $H_E$ in $H$.

(c) Let $P : H \to H$ be the linear map given by $P(f) = f\chi_E$. Calculate the norm of $P$.

# Complex Analysis Preliminary Examination

July 14, 2010

## 1 Theorems and Definitions

Answer **at least four** of the following questions. You are asked to give precise definitions or precise statements of theorems. **Explain every symbol you use in the context of your statement or definition.**

1. Let $U \subset \mathbb{C}$ be an open set, and let $\{f_n : U \to \mathbb{C}\}$ be a sequence of functions. Define what it means for the sequence $\{f_n\}$ to *converge locally uniformly* in $U$.

2. Let $f$ be analytic and non-constant in some disk $B(z_0, r) \subset \mathbb{C}$. Define the *multiplicity* of $f$ at $z_0$.

3. State the *Theorem of Casorati-Weierstrass.*

4. Define the *winding number* of a piecewise smooth closed path $\gamma$ in $\mathbb{C}$ around a point $z_0$.

5. State *Morera's Theorem.*

6. Let $D$ be a domain in the complex plane with $0 \notin D$. Define a *branch of the logarithm function in $D$.*

## 2 Problems

Solve **at least four** of the following problems. **Explain your reasoning and show all work.**

1. Calculate:

$$\lim_{r \to \infty} \int_{\gamma_r} \frac{e^{iz}}{z^2} dz,$$

   where for each $r$ the path $\gamma_r$ is defined by $\gamma_r(t) = re^{it},\ t \in [0, \pi]$.

2. This question has two parts.

   (a) Suppose that $f$ has a simple pole at $z = a$ and let $g$ be analytic in an open set containing $a$. Show that $\text{Res}(a, f \cdot g) = g(a)\text{Res}(a, f)$.

(b) Use part (a) to show that if $D$ is a domain and $f$ is analytic in $D$ except for simple poles at $a_1, \ldots, a_n$; and if $g$ is analytic in $D$ then

$$\frac{1}{2\pi i}\int_\gamma f(z)g(z)dz = \sum_{k=1}^{n} n(\gamma, a_k)g(a_k)Res(a_k, f)$$

for any piecewise smooth closed path $\gamma$ not passing through $a, \ldots, a_n$ and such that $\gamma \sim 0$ in $D$.

3. Find all possible values of

$$\int_\gamma \frac{dz}{1+z^2},$$

where $\gamma$ is any closed piecewise smooth path in $\mathbb{C}$ not passing through $\pm i$.

4. Let $D$ be a domain in $\mathbb{C}$, and let $f : D \to \mathbb{C}$ be analytic and injective. Show that $f'(z) \neq 0$ for all $z \in D$.

5. Let $R > 0$ and $D = \{z : |z| > R\}$. An analytic function $f : D \to \mathbb{C}$ has a removable singularity, a pole, or an essential singularity at infinity if the function $g(w) = f(\frac{1}{w})$ has, respectively, a removable singularity, a pole, or an essential singularity at $w = 0$. If $f$ has a pole at $\infty$ then the order of the pole is the order of the pole of $g$ at $w = 0$.

   (a) Prove that an entire function has a removable singularity at infinity if and only if it is constant.

   (b) Prove that an entire function has a pole at infinity of order $m$ if and only if it is a polynomial of degree $m$.

6. Suppose that a function $f$ is analytic in the unit disk $B(0,1)$, that $f(0) = f'(0) = 0$, and that $|f'(z)| \leq 1$ for every $z \in B(0,1)$. Prove that $|f(z)| \leq |z|^2/2$ for every $z \in B(0,1)$. For which functions $f$ can equality hold in this estimate at some $z \neq 0$?

Name: ______________________________

# COMPLEX ANALYSIS SECTION OF THE ANALYSIS QUALIFIER

WESLEYAN UNIVERSITY
June 11 2009

## INSTRUCTIONS

1. This exam consists of three sections. Answer **all** of the questions (short answer/definition) in the **first section** and the **second section**. **Choose** one question to answer from the third section.

2. Clearly indicate the work you wish to be graded. **Show all work.** Good luck on the exam.

## 1. Definition/ Short Answer

**INSTRUCTIONS:** Answer **all** of the following questions. Be as complete as possible, and show all relevant work.

(1) Assume $U \subseteq \mathbb{C}$ is open. Define what it means for $f : U \to \mathbb{C}$ to be *analytic.*

(2) Let $U$ be a set in the complex plane, and let

$$\alpha_i : [0, 1] \to U, \ i = 1, 2$$

be a pair of closed paths in $U$. *Define* what it means for $\alpha_1$ to be *freely homotopic* to $\alpha_2$.

(3) State *Liouville's Theorem.*

(4) *True or False*: Satisfying the Cauchy-Riemann equations in a domain $D$ is *sufficient* to observe that a function $f$ is analytic on $D$.

(5) Let $U \subseteq \mathbb{C}$ be an open set in the domain of some complex-valued function $f$. *Define* what it means for $F$ to be a *primitive* of $f$.

(6) Let $U$ be an open set in $\mathbb{C}$, and let $\sigma$ be a cycle in $U$. *Define* what it means for $\sigma$ to be *homologous to zero* in $U$.

(7) Let $D \subset \mathbb{C}$ be the bounded region inside

$$\{z \in \mathbb{C} : |Re(z)| + |Im(z)| = 1\},$$

where $z = x + iy$.

Orient $\partial D$ positively. *Calculate*

$$\int_{\partial D} \frac{\cos z}{z(z^2+8)} dz.$$

**Show** all work.

## 2. Section II

**INSTRUCTIONS:** Answer **both** of the following questions. Be as **complete** as possible, e.g. **fully quote** statements that you are using, and **show** all relevant work.

(1) Let $f$ be an entire function that satisfies the estimate $|f(z)| \leq m\, e^{\alpha\, x}$, where $m$ and $\alpha$ are positive constants and $z = x + iy$.

Show $f(z) = Ae^{\alpha\, z}$ for some complex constant $A$.

(2) Let $D$ be a domain in $\mathbb{C}$. If every function $f$ which is analytic in $D$ possesses a primitive in $D$, *show* that $D$ is *simply connected.* (Recall that a domain is simply connected if every cycle in the domain is homologous to zero.)

## 3. Section III

**INSTRUCTIONS:** Answer **one** of the following questions. Be as **complete** as possible, and **show** all relevant work. **Clearly indicate** the problem which is to be graded.

(1) Let $f(z)$ be analytic in the disk $\Delta = \Delta(z_0, r)$ $(r > 0)$, and assume that there exists a $m > 0$ so that

$$|f(z) - f(z_0)| \leq m$$

for all $z \in \Delta$.

*Prove:* The derivative $f'$ satisfies the inequality

$$|f'(z_0)| \leq \frac{m}{r}$$

on $\Delta$.

(2) Let $u : D \to \mathbb{R}$, where $D$ is a domain. The function $u \in C^2(D)$ is *harmonic* if

$$u_{xx} + u_{yy} = 0.$$

A function $v : D \to \mathbb{R}$ is a *harmonic conjugate* to $u$ if the function $f = u + iv$ is analytic in $D$.

*Prove:* If $D$ is simply connected then $u$ possesses a harmonic conjugate. (**Hint:** consider the function $g = u_x - iu_y$.)

# Real Analysis Prelim, Summer 2009

1. Give a definition of each of the following:

   Dynkin system (in $\Omega$)

   outer measure (in $\Omega$)

   set which is $\mu^*$-measurable (if $\mu^*$ is an outer measure in $\Omega$)

   integrable function (on the measure space $(\Omega, \mathcal{A}, \mu)$)

   absolute continuity of $\mu$ with respect to $\nu$

   normed vector space

2. State the following theorems:

   Uniqueness Theorem (concerning premeasures on rings)

   Dominated Convergence Theorem (due to H. Lebesgue)

   Hahn-Banach Theorem

   Uniform Boundedness Theorem (aka Banach-Steinhaus Theorem)

3. Provide solutions to the following:

a. If $(\Omega, \mathcal{A}, \mu)$ is a measure space with $\mu(\Omega) < \infty$ and if $A_1, A_2, \cdots \in \mathcal{A}$, prove that $\limsup \mu(A_n) \leq \mu(\limsup A_n)$.

b. It is always true that a content on a ring is subadditive. Assuming this without proof, show that a premeasure is always $\sigma$-subadditive.

c. Prove that if $f : X \to Y$ is a bijective (i.e. one-to-one and onto) linear mapping from the vector space $X$ to the vector space $Y$, then the inverse $f^{-1}$ of $f$ is also linear. Using a course theorem, if furthermore the spaces are normed and $f$ is continuous, when can one say that the inverse is continuous? Explain why.

d. In the Banach space of continuous real-valued functions on $[-1, 2]$ provided with their uniform norms, determine the uniform closure of the set of functions

$$\{f(x) = \sum_{n=0}^{N} a_n x^{3n+1} : N \geq 0, a_n \text{ real for } 0 \leq n \leq N\}.$$

# Complex Analysis Preliminary Examination

July 8, 2008

## 1 Theorems and Definitions

In the following questions you are asked to give precise definitions or precise statements of theorems. **Explain every symbol you use in the context of your statement.**

1. Let $f : U \to \mathbb{C}$ be a function that is analytic at a point $z_0 \in U$. Define the *multiplicity* of $f$ at $z_0$.
2. Let $\{f_n\}$ be a sequence of complex-valued functions defined on an open set $U \subset \mathbb{C}$, and let $f : U \to \mathbb{C}$. Define what it means for $\{f_n\}$ to *converge normally* (or *locally uniformly*) to $f$.
3. State *Liouville's Theorem.*
4. State the *Cauchy Integral Formula* in an open disk.

## 2 Problems

Solve **at least four** of the following problems. **Explain your reasoning and show all work.**

1. Find the Laurent series for $f(z) = \dfrac{1}{(z^2+1)^2}$, centered at $i$. Where does this series converge?
2. Calculate the following integrals:

   (a)
   $$\int_{|z-i|=10} \left(z + \frac{1}{z}\right)^4 dz$$

   (b)
   $$\int_{|z|=4} \left(\frac{\sin z}{z}\right)^{100} dz$$

3. Let $f$ be an analytic function in a domain $D$, and suppose that there exists a point $z_0 \in D$ such that $|f(z)| \geq |f(z_0)| > 0$ for all $z \in D$. Show that $f$ must be constant in $D$.

4. Let $D \subset \mathbb{C}$ be a domain and let $f, g : D \to \mathbb{C}$ be analytic. Let $z_0 \in D$ and suppose that $f(z_0) = g(z_0) = 0$. Suppose furthermore that

$$\lim_{z \to z_0} \frac{f'(z)}{g'(z)} \in \mathbb{C}$$

exists. Show that

$$\lim_{z \to z_0} \frac{f(z)}{g(z)}$$

exists as well and calculate this limit.

5. Let $U \subset \mathbb{C}$ be open, and let $f, g : U \to \mathbb{C}$ be analytic except for isolated singularities. Let $z_0 \in U$.

   (a) Is the residue of $f + g$ at $z_0$ equal to the residue of $f$ plus the residue of $g$?

   (b) Is the residue of $fg$ at $z_0$ equal to the product of the residues of $f$ and $g$?

# Real Analysis Preliminary Examination

July 8, 2008

## 1 Theorems and Definitions

Give precise statements of the following.

1. The definition of the Borel sets in the real line.
2. The definition of Lebesgue outer measure on the real line.
3. The Lebesgue monotone convergence theorem (in an abstract measure space).
4. The Hölder inequality (which concerns functions in certain $L^p$ spaces).
5. The Radon-Nikodym theorem.

## 2 Problems

1. Let $(X, \mathcal{B}, \mu)$ be a measure space and $f : X \to \mathbb{R}$ an integrable function. Prove that $\{x \in X \mid f(x) \neq 0\}$ is a countable union of sets of finite measure.
2. (a) Give an example of a sequence of real valued Lebesgue measurable functions on $[0,1]$ that converges to 0 almost everywhere (with respect to Lebesgue measure), but does not converge in $L^1$.

   (b) Give an example of a sequence of real valued Lebesgue integrable functions on $[0,1]$ that converges to 0 in $L^1$ (with respect to Lebesgue measure), but not.almost everywhere.
3. Suppose that $A \subset \mathbb{R}$ is a Lebesgue measurable set with the property that for every rational number $r$,

   $$A + r = A$$

   (The notation means: $A + r = \{a + r \mid a \in A\}$). Prove that either $\lambda(A) = 0$ or $\lambda(\mathbb{R} \backslash A) = 0$, where $\lambda$ denotes Lebesgue measure.
4. Suppose that $(X, \mathcal{B}, \mu)$ is a measure space with $\mu(X) < \infty$. Show that a sequence $\{f_n\}_{n=1}^{\infty}$ of measurable functions on $X$ converges to 0 in measure if and only if

   $$\lim_{n \to \infty} \int_X \frac{|f|}{1 + |f|} d\mu = 0.$$
5. Suppose that $(V, \| \ \|)$ is a real normed vector space in which every absolutely convergent series converges. Prove that $(V, \| \ \|)$ is complete with respect to its norm.