BASIC EXAM, SPRING 2010

Instructions: Do any 10 of the following questions. If you attempt more than 10 questions, indicate which one you would like to be considered for credit by crossing question we should not check (otherwise the first 10 will be taken). All problems worth 10 points. Parts of the problem do not carry equal weight. Little or no credit will be given for answers without adequate justification. Write your university identification number at the top of each sheet of paper.

Good luck!

Problem 1: Let $u_1, \ldots, u_n$ be orthonormal basis of $\mathbb{R}^n$ and let $y_1, \ldots, y_n$ be a collection of vectors in $\mathbb{R}^n$ satisfying $\sum_i ||y_i||^2 < 1$. Prove that vectors $u_1 + y_1, \ldots, u_n + y_n$ are linearly independent.

Problem 2: Let A be $n \times n$ real symmetric matrix and let $\lambda_1 \geq \ldots \geq \lambda_n$ be the eigenvalues of A. Prove that

$$\lambda_k = \max_{U, \dim U = k} \ \min_{x \in U, ||x|| = 1} \ < Ax, x >,$$

where $< \cdot, \cdot >$ denotes the usual scalar product in $\mathbb{R}^n$ and the maximum is taken over all k dimensional subspaces of $\mathbb{R}^n$

Problem 3: Let S and T be two normal transformations in the complex finite dimensional vector space V with a positive definite Hermitian inner product such that $ST = TS$. Prove that S and T have joint basis of eigenvectors.

Problem 4: (i). Let $A = (a_{i,j})$ be $n \times n$ real symmetric matrix such that $\sum_{i,j} a_{i,j} x_i x_j \leq 0$ for every vector $(x_1, \ldots, x_n)$ in $\mathbb{R}^n$. Prove that if $tr(A) = 0$ then $A = 0$.
(ii). Let T be a linear transformation in the complex finite dimensional vector space V with a positive definite Hermitian inner product. Suppose that $TT^* = 4T - 3I$, where I is identity transformation. Prove that T is positive definite Hermitian and find all possible eigenvalues of T.

Problem 5: Let A, B two $n \times n$ complex matrices which have the same minimal polynomial $M(t)$ and the same characteristic polynomial $P(t) = (t - \lambda_1)^{a_1} \cdots (t - \lambda_k)^{a_k}$, where $\lambda_i \neq \lambda_j$ for $i \neq j$. Prove that if $P(t)/M(t) = (t - \lambda_1) \cdots (t - \lambda_k)$, then these matrices are similar.

Problem 6: Let $A = \begin{pmatrix} 4 & -4 \\ 1 & 0 \end{pmatrix}$.
(i). Find Jordan form J of A and a matrix P such that $P^{-1}AP = J$.
(ii). Compute A^{100} and J^{100}.
(iii). Find a formula for a_n, when $a_{n+1} = 4a_n - 4a_{n-1}$ and $a_0 = a, a_1 = b$.

Problem 7: Let $\{f_n\}$ be a sequence of real-valued functions on the line, and assume that there is a $B < \infty$ such that $|f_n(x)| \leq B$ for all n and x. Prove that there is a subsequence $\{f_{n_k}\}$ such that $\lim_{k \to \infty} f_{n_k}(r)$ exists for all rational numbers r.

Problem 8: Assume that K is a closed subset of a complete metric space (χ, d) with the property that, for any $\epsilon > 0$, K can be covered by a finite number of sets $B_\epsilon(x)$, where

$$B_\epsilon(x) = \{y \in \chi : d(x,y) < \epsilon\}.$$

Prove that K is compact.

Problem 9: Assume that $f(x, y, z)$ is a real valued, continuously differentiable function such that $f(x_0, y_0, z_0) = 0$. If $\vec{\nabla} f(x_0, y_0, z_0) \neq \vec{0}$, show that there is a differentiable surface, given parametrically by $(x(s,t), y(s,t), z(s,t))$ with $(x(0,0), y(0,0), z(0,0)) = (x_0, y_0, z_0)$, on which $f = 0$.

Problem 10: Let $f(x,y)$ be the function defined by

$$f(x,y) = \frac{xy}{\sqrt{x^2 + y^2}}$$

when $(x,y) \neq (0,0)$ with $f(0,0) = 0$.

(a) Compute the directional derivatives of $f(x,y)$ at (0,0) in all directions where they exist.

(b) Is $f(x,y)$ differentiable at (0,0)? Prove your answer.

Problem 11: Suppose $\sum_{n=1}^{\infty} |a_n| < \infty$. Let σ be a one-to-one mapping of $\mathbb{N}$ onto $\mathbb{N}$. The series $\sum_{n=1}^{\infty} a_{\sigma(n)}$ is called a "rearrangement" of $\sum_{n=1}^{\infty} a_n$. Prove that all rearrangements of $\sum_{n=1}^{\infty} a_n$ are convergent and have the same sum.

Problem 12: Assume that $\{f_n\}$ is a sequence of nonnegative continuous functions on $[0,1]$ such that $\lim_{n\to\infty} \int_0^1 f_n(x)dx = 0$. Is necessarily true that

(a) There is a B such that $f_n(x) \leq B$ for $x \in [0,1]$ for all n?

(b) There are points x_0 in [0,1] such that $\lim_{n\to\infty} f_n(x_0) = 0$?

Prove your answers.