# Basic Exam Fall 2011

**IMPORTANT.** Write your university identification number on the upper right corner of this page *and* of *each* sheet of paper you use. **Do not write your name anywhere on the exam.** Leave clear margin on the top left corner of each page for stapling.

**Test Instructions:** Do any 10 of the following 12 problems. If you attempt more than 10 problems, indicate which 10 you wish to be graded. If you do not indicate, the first 10 attempted problems will be graded. Each question is worth 10 points, but parts within questions may not have equal values. Credit is based on correct work shown which is used to solve the problem. No credit will be given for answers without detailed justification. Partial credit will be given but not for vague work. The exam lasts 4 hours.

**Problem Scores** (NG=not graded)

Problem 1 ___________

Problem 2 ___________

Problem 3 ___________

Problem 4 ___________

Problem 5 ___________

Problem 6 ___________

Problem 7 ___________

Problem 8 ___________

Problem 9 ___________

Problem 10 ___________

Problem 11 ___________

Problem 12 ___________

**Total:** ___________

**Problem 1.** Let $(X, d)$ be a compact metric space and let $f : X \to X$ be a map satisfying

$$d(f(x), f(y)) < d(x, y), \qquad \forall x, y \in X \text{ with } x \neq y.$$

Prove that there is a unique point $x \in X$ so that $f(x) = x$.

**Problem 2.** A function $f : \mathbb{R}^n \to \mathbb{R}$ is called *convex* if $f$ satisfies

$$f(\alpha x + (1-\alpha)y) \leq \alpha f(x) + (1-\alpha)f(y), \qquad \forall x, y \in \mathbb{R}^n, \quad 0 \leq \alpha \leq 1.$$

Assume that $f$ is continuously differentiable and that for some constant $c > 0$, the gradient $\nabla f$ satisfies

$$(\nabla f(x) - \nabla f(y)) \cdot (x - y) \geq c(x-y) \cdot (x-y), \qquad \forall x, y \in \mathbb{R}^n,$$

where $\cdot$ denotes the dot product. Show that $f$ is convex.

**Problem 3.** Prove that the set of real numbers can be written as the union of uncountably many pairwise disjoint subsets, each of which is uncountable.

**Problem 4.** If you rearrange the order of terms in a sum $\sum a_n$ , sometimes you can change the limiting values. Find all the resulting limiting values of the following series. Prove your assertions.

1. $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$
2. $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$

**Problem 5.** Give an example of a function $f(x)$ on $[0, 1]$ with infinitely many discontinuities, but which is Riemann integrable. Include proof (don't just quote some theorem).

**Problem 6.** Let $f_n$ be a sequence of continuous functions on $[0, 1]$. Assume: (i) $f_n(x) \geq f_{n+1}(x)$ for all $x \in [0, 1]$; and (ii) $\lim_{n\to\infty} f_n(x) = 0$ for all $x \in [0, 1]$. Prove (don't just quote some theorem) that $f_n \to 0$ *uniformly* on $[0, 1]$.

**Problem 7.** Let $f : \mathbb{R} \to M_{n\times n}$ be a continuous function, where $M_{n\times n}$ is the space of $n \times n$ matrices. Show that the function $g(t) = \text{rank}(f(t))$ is lower semi-continuous, meaning that if a sequence $t_n$ converges to $t$ then $g(t) \leq \liminf_n g(t_n)$. Is $g$ always continuous?

**Problem 8.** Assume that a complex matrix A satisfies $\ker((A - \lambda I)) = \ker((A - \lambda I)^2)$ for all $\lambda \in \mathbb{C}$. Show from first principles (i.e. without using the theory of canonical forms) that $A$ must be diagonalizable.

**Problem 9.** Let $V$ be a finite dimensional inner product space, and let $L : V \to V$ be a self-adjoint linear operator. Let $\mu$ and $\epsilon$ be given. Suppose there is a unit vector $x \in V$ such that

$$\|L(x) - \mu x\| \leq \varepsilon.$$

Prove that $L$ has an eigenvalue $\lambda$ so that $|\lambda - \mu| \leq \varepsilon$.

**Problem 10.** Let $A$ be a $3 \times 3$ real matrix with $A^3 = I$. Show that $A$ is similar to a matrix of the form

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix}$$

for some (real) $\theta$. What values of $\theta$ are possible?

**Problem 11.** (a) State and prove the rank-nullity theorem.

(b) Suppose $V$, $W$, and $U$ are finite dimensional vector spaces over $\mathbb{R}$ and that $T: V \to W$ and $S: W \to U$ are linear operators. Suppose further that $T$ is one-to-one, $S$ is onto, and $S \circ T = 0$. Prove that $\ker(S) \supseteq \operatorname{image}(T)$ and that $-\dim(V) + \dim(W) - \dim(U) = \dim(\ker(S)/\operatorname{image}(T))$.

**Problem 12.** Let $A$ be an $m \times n$ real matrix, and let $b \in \mathbb{R}^m$. Suppose $Ax$ and $Ay$ are both of minimal distance to $b$ (minimizing among members of image($A$)). Prove that $x - y \in \ker(A)$.