Basic Examination – Spring 2013

You may only submit solutions to **10** of the following **12** problems. Please indicate clearly which ones you want to submit.

1. (a) Define what it means for a function on [0,1] to be Riemann integrable.
(b) Show that every monotone increasing function on [0,1] is Riemann integrable.
(c) Use part (b) to give an example of a Riemann integrable function on [0,1] which has infinitely many discontinuities.

2. The approximation from "Simpson's Rule" for $\int_a^b f(x)dx$ is

$$S_{[a,b]}f = [\frac{2}{3}f(\frac{a+b}{2}) + \frac{1}{3}(\frac{f(a)+f(b)}{2})](b-a).$$

If f has continuous derivatives up to order three, prove that $|\int_a^b f(x)dx - S_{[a,b]}f| \leq C(b-a)^4 \max_{[a,b]} |f^{(3)}(x)|$, where C does not depend on f.

3. Prove that a metric space is sequentially compact if and only if it is complete and totally bounded. A metric space is totally bounded when for every $\epsilon > 0$ it can be covered by a finite number of balls of radius ϵ.

4. Denote by h_n the n-th harmonic number:

$$h_n = 1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n}$$

Prove that there is a limit

$$\gamma = \lim_{n\to\infty} (h_n - \ln n)$$

5. Define polynomials $U_n(x)$, $n = 0, 1, 2, ..$ as follows:

$$U_1(x) = 1, \quad U_2(x) = 2x, \quad U_{n+1}(x) = 2xU_n(x) - U_{n-1}(x)$$

(a) Prove that

$$U_n(\cos\theta) = \frac{\sin((n+1)\theta))}{\sin(\theta)}.$$

(b) Prove that the polynomials $U_n(x)$ satisfy:

$$\int_{-1}^{1} U_m(x)U_n(x)\sqrt{1-x^2}dx = \begin{cases} 0 & \text{when } m \neq n \\ \pi/2 & \text{when } m = n \end{cases}$$

6. (a) Prove that diagonalizable matrices are dense in the set of all $n \times n$ matrices with complex entries.
(b) Are diagonalizable matrices with real entries dense in the set of all $n \times n$ matrices with real entries?

7. (a) Define the operator norm of a real $n \times n$ matrix (considered as a linear transformation from $\mathbb{R}^n$ to $\mathbb{R}^n$).

(b) Denote the $n \times n$ identity matrix by I. Show that the series $\exp(A) = I + A + A^2/2! + A^3/3! + \ldots$ converges to a limit in the usual sense of convergence of matrices (convergence entry by entry).

(c) Show that the series $\ln(I+A) = A - A^2/2 + A^3/3 + \cdots + (-1)^{n+1}A^n/n + \cdots$ converges if the operator norm of A is less than one.

(d) Show that $\exp(\ln(I+A)) = I + A$ if the operator norm of A is less than 1.

8. Let T be a linear transformation from a finite-dimensional vector space V with an inner product to a finite dimensional vector space W also with an inner product (the dimension of W can be different from the dimension of V here).

(a) Define the adjoint $T^* : W \to V$.

(b) Show that if matrices are written relative to orthonormal bases of V and W then the matrix of T^* is the transpose of the matrix of T.

(c) Show that the kernel (null space) of T^* is the orthogonal complement of the range of T.

(d) Use (b) and (c) to show that the row rank of a matrix is the same as the column rank of that same matrix.

9. Suppose A is a real $n \times n$ orthogonal matrix, i.e. the transpose of A is its inverse.

(a) Show that A is similar to a matrix which consists of 2×2 blocks down the diagonal along with some diagonal elements that are +1 or -1 with the 2×2 blocks being rotation matrices of the form

$$\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}.$$

(b) Use part (a) to show that if n is odd, then there is a nonzero vector v such that $A^2 v = v$.

10. Denote by G the set of real 4×4 upper triangular matrices with 1's on the diagonal. Fix

$$M = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Denote by C the set of matrices in G commuting with M.

(a) Prove that C is an affine subspace in the space $\mathbb{R}^{16}$ of all 4×4 real matrices. S is an "affine subspace" of a vector space V if there is a vector $w \in V$ such that $S^0 = \{v - w : v \in S\}$ is a subspace of V. The dimension of S is defined to be the dimension of S^0.

(b) Find the dimension of C.

11. Define the Fibonacci sequence F_n by $F_0 = 0$, $F_1 = 1$, and recursively $F_n = F_{n-1} + F_{n-2}$ for $n = 2, 3, 4, \ldots$

(a) Show that the limit as n goes to infinity of F_n/F_{n-1} exists and find its value.
(b) Prove that $F_{2n+1}F_{2n-1} - F_{2n}^2 = 1$ for all $n \geq 1$.

12. Define $\pi = 4\int_0^1 \frac{dx}{1+x^2}$. Prove that $\pi = 4(1 - 1/3 + 1/5 - 1/7 + \cdots$. Be careful to give a complete proof.