Warm-up Problems

- If X is T_2 and $A \subseteq X$ is compact, then A is closed.
- If X is compact and $A \subseteq X$ is closed, then A is compact.
- (June 2011, A1) If $f : X \to Y$ is continuous and $A \subseteq X$ is compact, then $f(A)$ is compact.

1. (Jan 2002, A1) Let X be a compact space and let
$$A_1 \supseteq A_2 \supseteq \cdots \supseteq A_n \supseteq \cdots$$
be a descending chain of nonempty closed subsets of X. Show that their intersection $\cap_{n=1}^{\infty} A_n$ is nonempty.

2. (June 2005, A2) Let X be the set of integers, let $\mathcal{C}_1 := \{A \subseteq X | X - A \text{ is finite}\}$, and let $\mathcal{C}_2 := \{A \subset X | 0 \notin A\}$. Show that the union $\mathcal{T} := \mathcal{C}_1 \cup \mathcal{C}_2$ is a topology on X, and show that this topological space is compact.

3. (Jan 2006, A4) Let $\mathcal{T}, \mathcal{T}'$ be two topologies on X. Show that if $(X, \mathcal{T})$ is compact and Hausdorff, $\mathcal{T} \subseteq \mathcal{T}'$, and $\mathcal{T} \neq \mathcal{T}'$, then $(X, \mathcal{T}')$ is Hausdorff but not compact.

4. (June 2008, A4) Let $Y = (0, 1) \times (0, 1)$ and let X be the set $X = Y \cup \{\star\}$. Let $\mathcal{T}_1 := \{U | U \text{ is a Euclidean open subset of } Y\}$, let $\mathcal{T}_2 := \{X - C | C \text{ is a compact subset of the Euclidean space } Y\}$, and let $\mathcal{T} := \mathcal{T}_1 \cup \mathcal{T}_2$. Then $(X, \mathcal{T})$ is a topological space called the *one-point compactification* of $(0, 1) \times (0, 1)$ (you don't need to prove this). Show that the space $(X, \mathcal{T})$ is compact.

5. (June 2009, A4) Let $(X, \mathcal{T})$ and $(X, \mathcal{T}')$ be topological spaces with $\mathcal{T} \subseteq \mathcal{T}'$. [(a) omitted]
(b) If $(X, \mathcal{T}')$ is compact, must $(X, \mathcal{T})$ also be compact?

6. (June 2010, A2) Let X be a Hausdorff space, $p \in X$, and $A \subseteq X$ a compact subset of X disjoint from p. Show that there exist disjoint open sets $U, V \subseteq X$ with $p \in U$ and $A \subseteq V$.

7. (June 2011, A4) Suppose A, B are disjoint, compact subspaces of the Hausdorff topological space X. Prove that there are open subsets U, V of X such that $A \subseteq U$, $B \subseteq V$, and $U \cap V = \emptyset$

8. (June 2004, A1; June 2007, A4) A space $(X, \mathcal{T})$ is called *locally compact* if, for every $x \in X$ and $x \in U \in \mathcal{T}$, there is a compact set $C \subseteq U$ containing an open neighborhood of x. Show that the Cartesian product of two locally compact spaces is locally compact.

9. Suppose that X is compact and Y is Hausdorff. Prove that every one-to-one, onto, continuous map $f : X \to Y$ is a homeomorphism.