Warm-up Problems

- X is a connected space if and only if X and the empty set are the only open and closed sets.
- If X is a path connected space, then X is connected.

1. (Jan 2002, A3) Let X be a topological space, and $A, B \subseteq X$ be connected subsets of X. Show that if $A \cap \overline{B} \neq \emptyset$, then $A \cup B$ is a connected subset of X.
2. (Jan 2002, A5) Show that $\mathbb{R}$ and $\mathbb{R}^2$ (with their usual topologies) are not homeomorphic.
3. (June 2004, A5) A topological space X is said to be *contractible* if the identity map $I_X : X \to X$ is homotopic to a constant map $c : X \to X$. Show that every contractible space is path-connected.
4. (June 2005, A3) Let X be a connected space with connected subset Y, and suppose that $X \backslash Y$ is not connected, with $X \backslash Y = A \cup B$ a separation of $X \backslash Y$. Show that if B is open in X, then $A \cup Y$ is a connected subset of X.
5. (Jan 2006, A5) Recall that a topological space X is *locally connected* if for every point $x \in X$ and every neighborhood U of x there exists a connected neighborhood V of x with $V \subseteq U$.
   (a) Prove that a topological space X is locally connected iff for every open set $U \subseteq X$ the components of U are open.
   (b) Now let $p : X \to Y$ be a quotient map. Prove that if C is a component of an open subset $V \subseteq Y$ then $p^{-1}(C)$ is a union of components of $p^{-1}(V)$.
   (c) Deduce that if X is locally connected then so is Y.
6. (June 2007, A2) A space $(X, \mathcal{T})$ is called *locally path-connected* if, for every $x \in X$, every neighborhood of x contains a path-connected neighborhood of x. Show that a connected, locally path-connected space is path-connected.
7. (Jan 2008, B7) Show that if A is a proper subset of a connected space X and B is a proper subset of a connected space Y, then $(X \times Y) \backslash (A \times B)$ is connected.
8. (June 2008, A1) Let $p : X \to Y$ be a quotient map. Show that if Y is connected and moreover each set $p^{-1}(\{y\})$ is connected, then X is connected.
9. (June 2010, A4) Let $Z = X \cup Y$ for X and Y connected subspaces of Z with $X \cap Y = \emptyset$. Let $x_0 \in X$ and $y_0 \in Y$. Let $\sim$ be the equivalence relation generated by the equivalence $x_0 \sim y_0$. Show that the quotient space $Z/\sim$ is connected.
10. (May 2015, A2) If $X = \mathbb{R}^n$ with the usual Euclidean topology and $U \subseteq X$ is an open, connected subspace of X, show that U is also path-connected.