Warm-up Problems

- $T_4 \Rightarrow T_3 \Rightarrow T_2 \Rightarrow T_1 \Rightarrow T_0$
- (June 2004, A3) Show that if X is a Hausdorff space, and $A, B \subseteq X$ are disjoint, finite subsets of X, then there are disjoint open sets U, V in X with $A \subseteq U$ and $B \subseteq V$.
- (June 2009, A4) Let $(X, \mathcal{T})$ and $(X, \mathcal{T}')$ be topological spaces with $\mathcal{T} \subseteq \mathcal{T}'$). (a) If $(X, \mathcal{T}')$ is normal, must $(X, \mathcal{T})$ also be normal?

Separation Axioms:

1. (Jan 2004, A2) Show that if the metric space (X, d) is separable (ie, it contains a countable dense subset), then the metric topology on X is second countable.
2. (Jan 2004, A4) Two subsets $A, B \subseteq X$ of the space $(X, \mathcal{T})$ are called *separated* if there are $U, V \in \mathcal{T}$ with $A \subseteq U \subseteq X \backslash B$ and $B \subseteq V \subseteq X \backslash A$. X is called *completely normal* if X is T_1 and for every pair of separated subsets A, B there are $U, V \in \mathcal{T}$ such that $A \subseteq U$, $B \subseteq V$, and $U \cap V = \emptyset$.

   Show that a space $(X, \mathcal{T})$ is completely normal if and only if every subset of X is normal.
3. (June 2004, A4) Show that for a topological space X, if every $x \in X$ has an open neighborhood whose closure is a regular space, then X is regular.
4. (June 2009, A1) Show that every compact metrizable space has a countable basis.

Counterexamples:

1. (Jan 2004, A3) Let $(X, \mathcal{T})$ be a Hausdorff space and let $\mathcal{T}' = \{U \subseteq X : X \backslash U \text{ is compact}\} \cup \{\emptyset\}$. Show that $\mathcal{T}'$ is a topology on X, and is coarser than $\mathcal{T}$. Show that, in general, they need not be equal.
2. (June 2008, A1) Let A be a subset of a topological space X and let B be a subset of a topological space Y. Let $X \times Y$ be the product space, and let $\text{int}_Z(C)$ denote the interior of the set C in the space Z. Prove or give a counterexample to $\text{int}_{X \times Y}(A \times B) = \text{int}_X(A) \times \text{int}_Y(B)$.
3. (June 2009, A2) For a topological space X and $y \in X$, the *path component* P_y of X containing y is the largest path-connected subset with $y \in P_y \subseteq X$.
   (a) Show that this concept is well-defined (that is, show that every point y is contained in a largest path-connected subset).
   (b) Give an example of a space and a point $y \in X$ so that P_y is neither an open nor a closed subset of X.

Definition Problems:

1. (Jan 2002, A2) Let X be a topological space. A set $A \subseteq X$ is called *nowhere dense* if the closure $\overline{A}$ of A has empty interior, ie, $\text{int}(\overline{A}) = \emptyset$. Show that if $U \subseteq X$ is open, then $A = \overline{U} \backslash U$ is nowhere dense.

2. (June 2005, A4) A topological space X is called *metacompact* if for every open cover C of X, there is a subcover C' satisfying the property that for every point $p \in X$, there are only finitely many open sets in C' containing p.
   (a) Show that metacompactness is a homeomorphism invariant.
   (b) Let X be the integers with the topology $\mathcal{T} := \{U \subseteq X | 0 \in U\} \cup \{\emptyset\}$. Show that this space is not metacompact.
3. (Jan 2006, A2) A topological space $(X, \mathcal{T})$ is called *limit-point compact* if every infinite subset A of X has a limit point. Show that every closed subset of a limit-point compact space is limit-point compact.
4. (June 2014, A4) A space X is *locally connected* if for each $x \in X$ and each open set U containing x, there is a connected open set V in X satisfying $x \in V \subseteq U$. Let X and Y be locally connected spaces. Determine whether or not the product space $X \times Y$ must also be locally connected.

Separation Axioms Definitions

- If a space X has a countable basis for its topology, then X is said to be *second countable*.
- A subset A of a space X is *dense* in X if $\overline{A} = X$.
- A space having a countable dense subset is *separable*.
- X is T_0 if for any two distinct points $a, b \in X$ there is an open set U containing one of a or b but not both.
- X is T_1 if for any two distinct points $a, b \in X$ there are open sets U, V in X with $a \in U, b \notin U$, and $b \in V, a \notin V$.
- X is T_2 (Hausdorff) if for any two distinct points $a, b \in X$ there are disjoint open sets U, V in X with $a \in U$ and $b \in V$.
- X is T_3 (Regular) if X is T_1 and for any point $a \in X$ and closed set B in X with $a \notin B$, there are disjoint open sets U, V in X with $a \in U$ and $B \subseteq V$.
- X is T_4 (Normal) if X is T_1 and for any two disjoint closed sets A, B in X there are disjoint open sets U, V in X with $A \subseteq U$ and $B \subseteq V$.

Useful examples:

- $\mathbb{R}$ with the finite complement topology is T_1 but not T_2.
- $X = \{a, b\}$ with $\mathcal{T} = \{\emptyset, \{a\}, X\}$ is T_0 but not T_1
- Any space X with the power set (discrete) topology is T_4.

Common places to look for counterexamples

- Flea and Comb (differentiates connected and path connected)
- Comb (sans the flea) (differentiates path connected and locally path connected)
- Topologist's sine curve (differentiates connected and path connected)
- 2 or 3 point sets (especially useful if you assume your space must be compact)
- Any space with the discrete topology
- Any space with the indiscrete topology