Warm-up Problems

- Let $A \subseteq X$ and $r : X \to A$ a retraction. Prove that $r_* : \pi_1(X) \to \pi_1(A)$ is a surjection.
- Let $B \subseteq Y$ and let $p : Y \to B$ be a deformation retraction. Prove that $p_* : \pi_1(Y) \to \pi_1(B)$ is an isomorphism.

1. (Jan 2002, B1) Show that if X is path-connected, locally path-connected space, with $\pi_1(X)$ finite, then every continuous map $f : X \to S^1$ is homotopic to a constant map.
2. (Jan 2002, B3) Let X be the 2-sphere, and $A \subseteq X$ the equitorial circle in X. Show that there is no retraction $r : X \to A$.
3. (June 2014, A2)
   (a) Let $q : A \to B$ be continuous and $i : I \to I$ be the identity function. Define the function $q \times i : A \times I \to B \times I$ by $(q \times i)(a, t) := (q(a), i(t))$. Show that $q \times i$ is continuous.
   (b) (You may assume the fact: If $q : A \to B$ is a quotient map, then so is $q \times i$.) Let X be a path-connected Hausdorff space. The *cone* CX of X is the quotient of the space $X \times I$ by the smallest equivalence relation such that $(p, 0) \sim (q, 0)$ for all $p, q \in X$. Prove that there is a deformation retraction from CX to the singleton subspace $\{[(p, 0)]\}$.
4. (May 2015, A4) Let $f : X \to Y$ be a continuous map, and suppose that there are continuous maps $g, h : Y \to X$ with $f \circ g \simeq \mathrm{Id}_Y$ and $h \circ f \simeq \mathrm{Id}_X$.
   (a) Show that $g \simeq h$.
   (b) Show, moreover, that f is a homotopy equivalence.
5. (Arizona Spring 2005) Let $f : S^1 \vee S^1 \to T$ be a continuous map, where T is the torus. Show that there is no continuous map $g : T \to S^1 \vee S^1$ such that $f \circ g$ is the identity map on T.
6. (June 2004, A2) Let X be the unit sphere in $\mathbb{R}^3$, and define an equivalence relation on X by
   $$(x, y, z) \sim (x', y', z') \text{ if and only if } z = z'.$$
   Let $Z = X/\sim$ be the quotient space under this equivalence relation, with the quotient topology. Show that Z is homeomorphic to the interval $[-1, 1]$. (Here $\mathbb{R}^3$ and $[-1, 1]$ have their usual topologies, and X has the subspace topology.)
7. (June 2008, A3)
   (a) Prove that a map $f : X \to Y$ between topological spaces is continuous if and only if $f(\overline{A}) \subseteq \overline{f(A)}$ for all $A \subseteq X$. [Here $\overline{A}$ denotes the closure of A.]
   (b) Prove that if f is continuous and $f(\overline{A})$ is closed for some $A \subseteq X$ then $f(\overline{A}) = \overline{f(A)}$.
8. (June 2009, A3) Let S^1 and D^2 be the unit circle and closed unit disk in the (Euclidean) space $\mathbb{R}^2$. Let $\sim$ be the smallest equivalence relation on the product space $S^1 \times D^2$ satisfying $(x, y) \sim (x, (0, 0))$ for all $x \in S^1$ and $y \in D^2$. Let $Z := (S^1 \times D^2)/\sim$ be the corresponding quotient space. Prove that Z is homeomorphic to S^1.