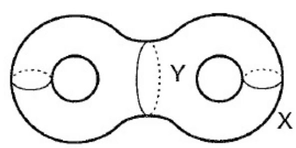
Warm-up Problems

- Let $G = \langle x, y | x^2yxy = 1\rangle$. Describe a space X that has $\pi_1(X) = G$.
- Find two spaces X and Y such that $\pi_1(X) \cong \pi_1(Y)$ and are nontrivial, and are NOT homeomorphic.
- Find $\pi_1(T^2)$, where $T^2 = S^1 \times S^1$ is the standard torus, in two different ways.

1. (Jan 2002, B2) Find the fundamental group of the space X consisting of $\mathbb{R}^3$ with the three coordinate axes removed.
2. (June 2005, B5) Find a cell structure for the quotient space obtained by identifying two distinct points a, b in a 2-torus to a third point c in a 2-sphere, and compute a presentation for the fundamental group of this space.
3. (June 2007, B3) Let X be the surface of genus 2 shown below and let $Y \supseteq X$ be the region of $\mathbb{R}^3$ that it encloses. Show that there is no retraction $r : Y \to X$ (that is, there is no continuous map for which $r(x) = x$ for all $x \in X$).



4. (June 2008, B5) Let X be the triangular parachute formed from the standard 2-simplex Δ^2 by identifying the three vertices with one another.
   (a) Compute a presentation for $\pi_1(X)$.
   (b) Show that $\pi_1(X)$ is isomorphic to a free group F_n for some n; what is n?
5. (June 2010, B5) Show that if A is a path-connected subspace of X, $x_0 \in A$, and the homomorphism $i_* : \pi_1(A, x_0) \to \pi_1(X, x_0)$ induced by the inclusion map $i : A \to X$ is surjective, then every path in X with endpoints in A is homotopic relative to the endpoints (ie, via a homotopy that fixes each endpoint) to a path in A.
6. Attach a 2-disk D^2 to a torus $S^1 \times S^1$ by the attaching map $e^{2\pi it} \mapsto (e^{2\pi it}, e^{2\pi it})$, thinking of the boundary ∂D^2 and each S^1 factor as the unit circle in the complex plane, and let X be the resulting space. Compute a presentation for the fundamental group of X.
7. (June 2011, B5) Let X be the space obtained by deleting three distinct points from $\mathbb{R}^2$. Compute $\pi_1(X)$.

Covering space bonus problems:

1. (June 2010, B7) Use covering space theory to show that if H is a subgroup of index 3 in a finitely presented group G, then H is finitely presented.
2. Let $F(a, b)$ be the free group of two generators. What is the presentation complex and the Cayley complex for $F(a, b)$? Find an index 4 subgroup and the covering space that corresponds to it.