Warm-up Problems

- Explicitly prove that $\mathbb{R}$ is the universal cover of S^1.
- Prove or disprove: $S^2 \vee S^2$ is a covering space of S^2.

1. (Jan 2002, B4) Let $p : \widetilde{X} \to X$ be a covering map, with X locally path-connected, and $x_0 \in X$. Show that for $y_0, y_1 \in p^{-1}(\{x_0\})$, there is a deck transformation of $\widetilde{X}$ taking y_0 to y_1 if and only if $p_*(\pi_1(\widetilde{X}, y_0)) = p_*(\pi_1(\widetilde{X}, y_1))$.
2. (June 2005, B6) Show, using covering spaces, that the fundamental group of the Klein bottle is not abelian.
3. (Jan 2006, B7) Let $p : \widetilde{X} \to X$ be a covering map, and $f : Y \to X$ be a continuous map. Define $\widetilde{Y} = \{(y, \widetilde{x}) \in Y \times \widetilde{X} : f(y) = p(\widetilde{x})\} \subseteq Y \times \widetilde{X}$, with the subspace topology inherited from $Y \times \widetilde{X}$, and define $q : \widetilde{Y} \to Y$ by $q(y, \widetilde{x}) = y$. Show that q is also a covering map.
4. (June 2007, B2) Let $p : (\widetilde{X}, y_0) \to (X, x_0)$ be a covering space projection, $x_0 \in A \subseteq X$, and $\iota : A \to X$ the inclusion map. Show that $q = p|_{p^{-1}(A)} : (p^{-1}(A), y_0) \to (A, x_0)$ is also a covering space, and $\ker(\iota_*) \subseteq \operatorname{im}(q_*) \subseteq \pi_1(A, x_0)$.
5. (June 2008, B6) Let $p_i : (\widetilde{X}_i \widetilde{x}_i) \to (X_i, x_i)$ be covering spaces for $i = 1, 2$.
   (a) Show that the product space $\widetilde{X}_1 \times \widetilde{X}_2$ together with the map $p : \widetilde{X}_1 \times \widetilde{X}_2 \to X_1 \times X_2$ defined by $p(y, z) := (p_1(y), p_2(z))$ is also a covering space.
   (b) Find the universal covering of the space $S^1 \times D^2 \times S^1$.
6. (June 2009, B6) Let $p : \widetilde{X} \to X$ be a covering space. Show that if X is Hausdorff, then $\widetilde{X}$ is also Hausdorff.
7. (June 2009, B7) Use covering space theory to prove that any finitely presented group G has only finitely many subgroups of index 3.
8. (June 2010, B7) Use covering space theory to prove that if H is a subgroup of index 3 in a finitely presented group G, then H is finitely presented. More generally, (June 2014, B7), show the result if H is only assumed to be of finite index.
9. (June 2014, B6) Let $p : \widetilde{X} \to X$ be a covering space with $\widetilde{X}$ path-connected. Let A be a subspace of X with inclusion map $i : A \hookrightarrow X$, and suppose that for $a \in A$ the induced homomorphism $i_* : \pi_1(A, a) \to \pi_1(X, a)$ is a surjection. Show that if $r, s \in \widetilde{X}$ and $p(r) = p(s) = a$, then there is a path from r to s in the subspace $p^{-1}(A)$ of $\widetilde{X}$.
10. (May 2015, B5) Suppose X, X_1, X_2 are path-connected spaces, $p : X_1 \to X$ and $q : X_2 \to X$ are (surjective) covering maps, and $f : X_1 \to X_2$ is a continuous map with $q \circ f = p$. Show that f is also a covering map.
11. (May 2015, B6) Show, using covering space theory, that a finitely presented group G has only finitely many distinct subgroups $H \leq G$ of index 4.